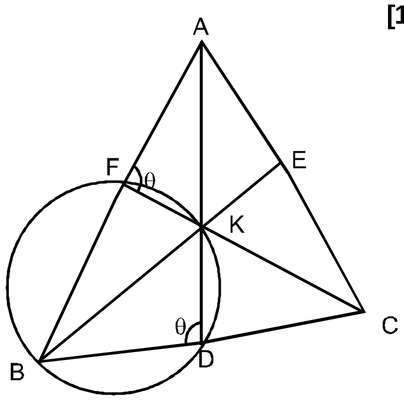
Regional Mathematical Olympiad-2011

Time : 3 hours **December 04, 2011**

Instruction :

- **Calculators (in any form) and protractors are not allowed.**
- **Rulers and compasses are allowed.**
- **Answer all the questions. Maximum marks : 100**
- **Answer to each question should start on a new page. Clearly indicate the questions number.**

1. Let ABC be a triangle. Let D, E, F be points respectively on the segments BC, CA, AB such that AD, BE, CE concur at the point K. Suppose BD / DC = BF / FA and $\angle$ ADB = $\angle$ AFC.
Prove that $\angle$ ABE = $\angle$ CAD. **[12]**



Sol. $\theta = \angle ADB = \angle AFC \Rightarrow \angle BFC = \pi - \theta$
$\Rightarrow$ BDKF is cyclic quadrilateral.
FK in chord of circle through B, D, K, F
$\Rightarrow \angle FBK = \angle FDK$ ...(1)
$\frac{BD}{DC} = \frac{BF}{FA} \Rightarrow \Delta$ CBA similar to ΔDBF
$\Rightarrow$ FD parallel to AC
$\Rightarrow \angle FDK = \angle DAC$
$\Rightarrow \angle ABE = \angle CAD$ Hence proved.

2. Let $(a_1, a_2, a_3 \ldots, a_{2011})$ be a permutation (that is a rearrangement) of the numbers 1, 2, 3 . . . , 2011. Show that there exist two numbers j, k. such that $1 \le j < k \le 2011$ and $|a_j - j| = |a_k - k|$. **[19]**

Sol. Total numbers $\rightarrow$ 2011
Total number of a's $\rightarrow$ 2011
difference $(a_i - i)$ may be from 0 to 2010.

Case-1 If difference is not zero
means $a_i - i \neq 0$
$a_i \neq i$
total differences = 2010
Numbers = 2011
Hence two numbers will have same difference
$\therefore |a_j - j| = |a_k - k|$

Case-2 If difference is zero

$a_i = i$

Any other $a_j = j$
then proved obviously

There exists no other j
such that $a_j = j$
Total a's $\rightarrow$ 2010
Total Numbers $\rightarrow$ 2010
Now difference may be from 1 to 2009
Hence two numbers will have same difference
$\therefore |a_j - j| = |a_k - k|$

3. A natural number n is chosen strictly between two consecutive perfect square. The smaller of these two squares is obtained by subtracting k from n and the larger one is obtained by adding ℓ to n. Prove that $n - k\ell$ is a perfect square. **[12]**

Sol. Let Squares are p^2 and $(p + 1)^2$

given $p^2 = n - k$

$(p + 1)^2 = n + \ell$

Now $n - k\ell$

$= n - (n - p^2)((p + 1)^2 - n)$

$= n - n(p + 1)^2 + n^2 + p^2(p + 1)^2 - np^2$

$= n - np^2 - 2np - n + n^2 - np^2 + p^2(p + 1)^2$

$= n^2 - 2np^2 - 2np + p^2(p + 1)^2$

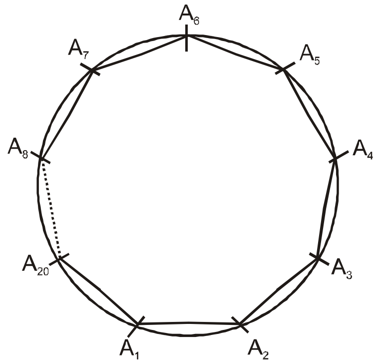
$= n^2 - 2np(p + 1) + p^2(p + 1)^2$

$= [n - (p + 1)p]^2$

Which is a perfect square.

4. Consider a 20-sided convex polygon K, with vertices $A_1, A_2, \ldots, A_{20}$ in that order. Find the number of ways in which three sides of K can be chosen so that every pair among them has at least two sides of K between them. (For example $(A_1A_2, A_4A_5, A_{11}A_{12})$ is an admissible triple while $(A_1A_2, A_4A_5, A_{19}A_{20})$ is not). **[19]**

Sol. Any side can be selected in ${}^{20}C_1$ ways



Let x, y, z are gapes between two sides and

$x \geq 2$, $y \geq 2$, $z \geq 2$

also $x + y + z = 17$

Let $x = t_1 + 2$, $y = t_2 + 2$, $z = t_3 + 2$

so $t_1 + t_2 + t_3 = 11$ where $t_1, t_2, t_3 \in W$

so total ways ${}^{11+3-1}C_{3-1} = {}^{13}C_2$

Now total required ways = $\dfrac{{}^{20}C_1 \times {}^{13}C_2}{3} = 520$

5. Let ABC be a triangle and let BB_1, CC_1 be respectively the bisectors of $\angle B$, $\angle C$ with B_1 on AC and C_1 on AB. Let E, F be the feet of perpendiculars drawn from A onto BB_1, CC_1 respectively. Suppose D is the point at which the incircle of ABC touches AB. Prove that AD = EF. **[19]**

Sol. Let radius of incircle = r

$\Rightarrow AI = r \operatorname{cosec} \dfrac{A}{2}$

$\angle BIC = \pi - \left(\dfrac{B}{2} + \dfrac{C}{2}\right) = \dfrac{\pi}{2} + \dfrac{A}{2} = \angle C_1IB_1$

$\Rightarrow \angle FAE = \dfrac{\pi}{2} - \dfrac{A}{2}$

If AI is diameter of circle then this circle passes through F & E and center of this circle is O

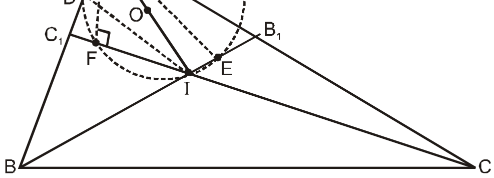
$\Rightarrow \angle FOE = \pi - A$

Now In the Δ FOE

$FE^2 = OF^2 + OE^2 - 2OF \cdot OF \cos(\pi - A)$

$= \dfrac{AI^2}{4} + \dfrac{AI^2}{4} + \dfrac{AI^2}{2}\cos A \qquad \left(OF = OE = \dfrac{AI}{2}\right)$

$= \dfrac{AI^2}{2}(1 + \cos A)$



$= AI^2 \cos^2 \frac{A}{2} \quad = r^2 \operatorname{cosec}^2 \frac{A}{2} \cos^2 \frac{A}{2}$

$= r^2 \cot^2 \frac{A}{2}$

$FE = r \cot \frac{A}{2}$ ..(1)

$ID = r$

In ΔADI $\quad \angle DAI = \frac{A}{2}$

$\Rightarrow AD = r \cot \frac{A}{2}$

$\Rightarrow AD = FE$

6. Find all pairs (x, y) of real numbers such that $16^{x^2+y} + 16^{x+y^2} = 1.$ **[19]**

Sol. Let two numbers are 16^{x^2+y}, 16^{x+y^2} use A.M. $\geq$ G.M.

$$\frac{16^{x^2+y} + 16^{x+y^2}}{2} \geq (16^{x^2+y}\ 16^{x+y^2})^{1/2}$$

$$\frac{1}{2} \geq (16^{x^2+x+y^2+y})^{1/2}$$

$$\Rightarrow 16^{x^2+x+y^2+y} \leq \frac{1}{4} \Rightarrow 4^{2(x^2+x+y+y^2)} \leq 4^{-1}$$

$2(x^2 + x + y^2 + y) \leq -1$

$(x + 1/2)^2 + (y + 1/2)^2 \leq 0$

which is possible only if

$x = -1/2,\ y = -1/2$