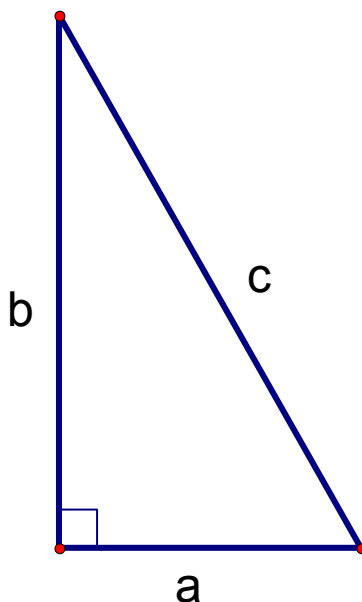# James Garfield's Proof of the Pythagorean Theorem

S. F. Ellermeyer
College Trigonometry (MATH 1112)
Kennesaw State University

The Pythagorean Theorem states that for any right triangle with sides of length a and b and hypotenuse of length c, it is true that $a^2 + b^2 = c^2$.

There are **many** different proofs of the Pythagorean Theorem. The proof that we will give here was discovered by James Garfield in 1876. Garfield later became the 20th President of the United States.

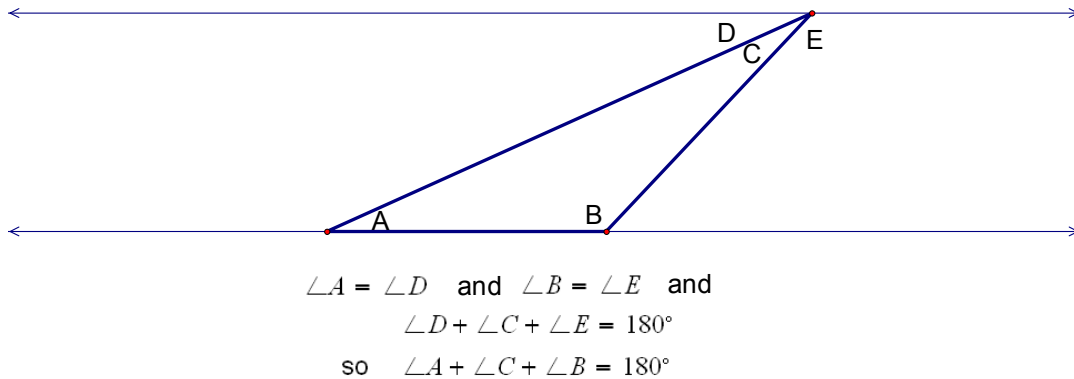The two key facts that are needed for Garfield's proof are:

1. The sum of the angles of any triangle is $180°$.
2. The area of a trapezoid with bases of length b_1 and b_2 and height h is
$$A = \frac{1}{2}(b_1 + b_2)h.$$

Before giving Garfield's Proof of the Pythagorean Theorem, we will first give proofs of the above two facts.

Proof of Fact 1

Let $\triangle ABC$ be any given triangle and draw parallel lines as shown in the figure below. The fact that $\angle A + \angle B + \angle C = 180°$ is deduced by using the fact that when parallel lines are cut by a transversal, the alternating interior angles are equal.

$$\angle A = \angle D \quad \text{and} \quad \angle B = \angle E \quad \text{and}$$
$$\angle D + \angle C + \angle E = 180°$$
$$\text{so} \quad \angle A + \angle C + \angle B = 180°$$

Proof of Fact 2

Recall that a trapezoid is a four–sided figure in which two opposite sides are parallel to each other. Suppose that the two parallel sides have lengths b_1 and b_2. The height, h, of the trapezoid is the distance between the two parallel sides.
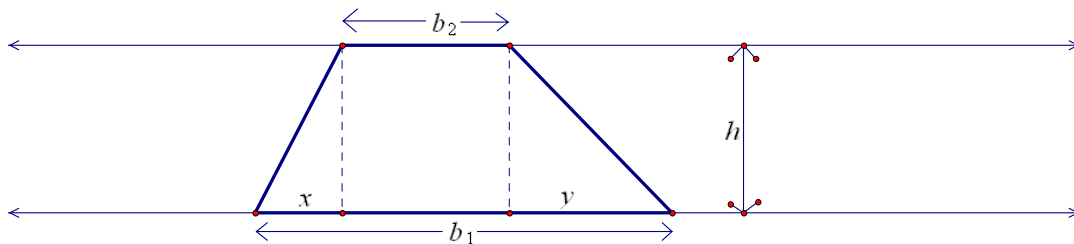
The trapezoid pictured above can be subdivided into a rectangle and two right triangles. Referring to the figure, we see that the area of the rectangle is

$$b_2 h,$$

the area of the triangle on the left is

$$\frac{1}{2}xh,$$

and the area of the triangle on the right is

$$\frac{1}{2}yh.$$

Thus the area of the trapezoid is

$$A = b_2 h + \frac{1}{2}xh + \frac{1}{2}yh$$
$$= \left(b_2 + \frac{1}{2}x + \frac{1}{2}y\right)h$$
$$= \left(\frac{1}{2}(x + y) + b_2\right)h.$$
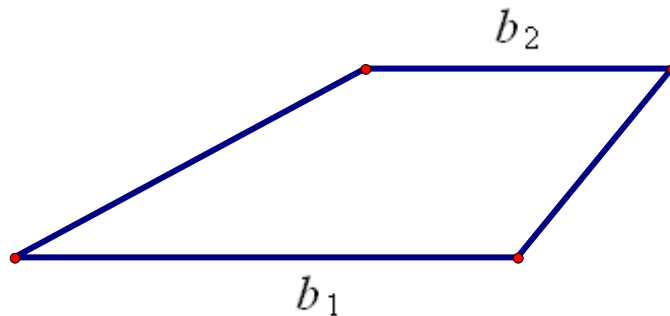
However, note that

$$b_2 + x + y = b_1$$

and thus

$$x + y = b_1 - b_2.$$

The area of the trapezoid is thus

$$
\begin{aligned}
A &= \left(\tfrac{1}{2}(b_1 - b_2) + b_2 \right) h \\
&= \left(\tfrac{1}{2}b_1 - \tfrac{1}{2}b_2 + b_2 \right) h \\
&= \left(\tfrac{1}{2}b_1 + \tfrac{1}{2}b_2 \right) h \\
&= \tfrac{1}{2}(b_1 + b_2)h.
\end{aligned}
$$

Note: In order to make this proof complete, there are a couple of other cases to be considered. For example, consider the case in which the trapezoid is as pictured below.


The case pictured above (and a third possible case) require slightly different proofs (which we invite the reader to carry out). In all cases we find that the area of a trapezoid is

$$A = \tfrac{1}{2}(b_1 + b_2)h.$$

Proof of the Pythagorean Theorem

In the figure shown below, we have taken an arbitrary right triangle with sides of length a and b and hypotenuse of length c and have drawn a second copy of this same triangle (positioned as pictured) and have then drawn an additional segment to form a trapezoid.
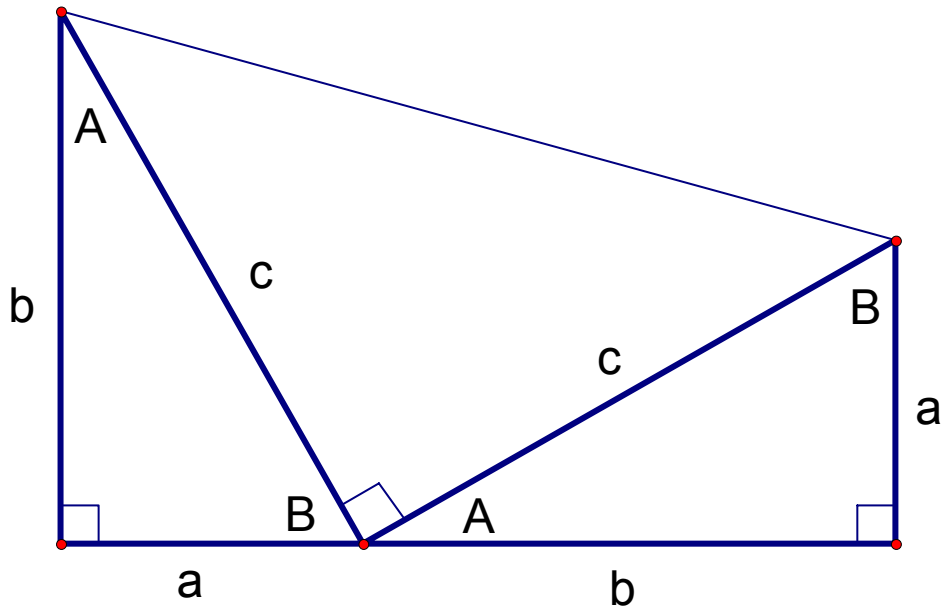
The parallel sides of the trapezoid (which are the left and right sides in the figure) have lengths a and b. The height of the trapezoid (which is the distance from left to right in the figure) is $a + b$. Thus the area of the trapezoid is

$$A = \frac{1}{2}(a + b)(a + b) = \frac{1}{2}(a + b)^2.$$

However, the area of the trapezoid is also the sum of the areas of the three triangles that make up the trapezoid. Note that the middle triangle is also a right triangle because $\angle A + \angle B = 90°$. The area of the trapezoid is thus

$$A = \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}cc = ab + \frac{1}{2}c^2.$$

We thus conclude that

$$\frac{1}{2}(a + b)^2 = ab + \frac{1}{2}c^2.$$

Multiplying both sides of this equation by 2 gives us

$$(a + b)^2 = 2ab + c^2.$$

Expanding the left hand side of the above equation then gives

$$a^2 + 2ab + b^2 = 2ab + c^2$$

from which we arrive at the conclusion that

$$a^2 + b^2 = c^2.$$