Selection Test - BIMC 2013

April 07, 2013 - 9.00 am to 11.30 am

PART A: SHORT ANSWER QUESTIONS

Write only the final answer in the answer sheet provided.

1. N identical balls ($4 \leq N \leq 2013$), numbered from 1 to N are placed in a box. Three balls are removed from the box; two of them have numbers which are not multiples of 3 while the third one has a number which is a multiple of 3. Now the probability of randomly choosing a ball with a multiple of 3 is lower than it was originally. How many different values can N take?

2. Let a and b be two *real* numbers such that $\frac{2014a+b}{a-b} = 2013$. What values can $\frac{a}{b}$ take?

3. What are the *integer* 4-tuples (a, b, c, d) which satisfy the equation, $a\sqrt{33} + b\sqrt{61} + c = d\sqrt{2013}$?

4. Let ABC be an isosceles triangle with base BC and let D be the midpoint of AC. Provided that BCD is also an isosceles triangle with base CD and $BC = 2$, what is the area of $\triangle ABC$?

5. The square $ABCD$ is drawn on a piece of paper. The paper is folded (along a straight line) so that B now coincides with the midpoint of DC. The side BC is divided by the fold into two segments of lengths a and b, with $a \leq b$. What is $\frac{b}{a}$?

6. A set S contains all the four-digit numbers, all of which after deleting *any* digit turn into a three-digit number that is a divisor of the original number. How many elements does S have?

7. What is the sum of all the integers n such that $n^2 + 5n + 1$ is a perfect square?

8. An 8 by 8 board needs to be filled with letters A, B, C and D so that two unit squares with a side or a vertex in common contain different letters, and so that letters A and B have the follwoing property; if A or B borders X horizontally or vertically (X can be A, B, C or D), then on the opposite side there is another X (unless it is at the edge of the board). How many different ways are there to fill the board with the letters?

PART B: ESSAY QUESTIONS

Full written solutions are required. Use the booklets provided to write the answers. Use separate booklets for each question.

1. Consider the numbers $1, 2, ..., n$. Find in terms of n, the largest integer t such that these numbers can be arranged in a row so that all consecutive terms differ by at least t. Justify your answer.

2. There are two bags containing balls. Initially there are m balls in one bag, and n in the other, where $m, n \geq 1$. Two different operations are allowed:

   A: Remove an equal number of balls from each bag;
   B: Double the number of balls in one bag.

   Is it always possible to empty both bags after a finite sequence of operations? Justify your answer.

   Operation B is now replaced with

   B': Triple the number of balls in one bag.

   Is it now possible to empty both bags after a finite sequence of operations? Justify your answer.

3. Let O be the circumcentre of $\triangle ABC$ and let $OBA'C, OCB'A$ and $OAC'B$ be rhombuses.

   Prove that the lines AA', BB' and CC' intersect at a point M. Also, prove that M, O and the circumcentre O' of $\triangle A'B'C'$ are collinear.

   What can be concluded about the triangle when O and M coincide?