G.C.E. (ADVANCED LEVEL) COMBINED MATHEMATICS
TEACHER TRAINING WORKSHOP – 2010

COMBINED MATHEMATICS
TEACHER TRAINING MANUAL – I

Department of Mathematics
Faculty of Science and Technology
National Institute of Education
Maharagama
Sri Lanka
2010

Preface

A training was given last year to grades 12 and 13 G.C.E. Advanced level Combined Mathematics teachers in schools island wide, on the new Combined Mathematics syllabus that was introduced in 2009. This training was conducted through 9 workshops under the direction of the Department of Mathematics, National Institute of Education, with the support of the relevant Provincial Education Departments and the participation of approximately 600 teachers.

The main aim of the Department of Mathematics, National Institute of Education, for 2010, is to conduct a series of 3 day training programmes, for a selected group of grades 12 and 13 G.C.E. Advanced level Combined Mathematics teachers in schools island wide, to help enhance the learning teaching process qualitatively. It was felt that this training would best serve those who have less than 10 years experience as Mathematics teachers, and a decision was taken to invite such teachers for the training programme.

A list of names of teachers who were identified as those who would benefit from this training programme was sent to us by each provincial office, and a selected group from this list was invited to the programme. For this 2010 teacher training workshop, 8 subject areas which were identified as areas that teachers needed to be updated on were selected and this journal has been prepared on them. We hope to conduct more training workshops in the near future that cover several other areas too. In this workshop attention will be paid to update teachers' knowledge on the relevant subject content under the Combined Mathematics syllabus, as well as to educate them on teaching and evaluation methods.

The workshop is carried out under the direction of the Department of Mathematics, NIE. Lectures will be conducted by University Lecturers.

The main aim of this training workshop is to help facilitate a productive learning teaching process in the class room for grades 12 and 13 students. It is hoped that this workshop will aid teachers in their efforts to promote mathematics among school children and bring their performance to a higher level. It is also hoped that both the training programme and the journal will support the teachers in their efforts to help students actualize the expected competencies and become citizens of good conduct and character.

Grades 12 and 13, Mathematics Project Team
2010

Team of Resource Persons

Grades 12 – 13 Curriculum Committee

Mr. Lal H. Wijesinghe	Director, National Institute of Education
Mr. K. Ganeshalingam	Chief Project Officer, National Institute of Education
Mr. G. P. H. Jagath Kumara	Project Officer, National Institute of Education
Ms. Iresha Ratnayake	Project Officer, National Institute of Education
Ms. M. Nilmini P. Peries	Project Officer, National Institute of Education
Mr. G. L. Karunarathne	Project Officer, National Institute of Education
Mr. S. Rajendram	Project Officer, National Institute of Education
Ms. Deepthi Gunawardene	Project Officer, National Institute of Education

Resource Persons for the lectures

Proffessor W.Ramasinghe	University of Colombo
Dr.Nishantha Perera	University of Colombo
Mr. Kapila de Silva	University of Sri Jayawardenepure
Dr.R.Sanjeewa	University of Sri Jayawardenepure
Dr. Sri Satkunarajah	University of Jaffna

Contribution from the external resource persons

Ms.Nilmini Abeydeera	Rtd. Chief project officer
Mr.T.H.G.Silva	Rtd. Teacher
Mr.S.A.Samaraweera	Rtd. Teacher
Ms. N.A.A.C.Nissanka	Rtd. Teacher
Ms.B.K.C.Perera	Teacher, Mahanama College, Colombo3

Translations

Dr. D. R. Jayawardhana	University of Colombo
Mr. N. R. Siriwardane	Rtd. Teacher

Content	Page Number

1.	Real Number	1
2.	Functions	10
3.	Polynomial and Rational Functions	19
4.	Quadratic Equations and Quadratic Functions	32
5.	Vectors	53
6.	Forces acting on a Particle	69
7.	Straight Lines	85
8.	Trigonometry	121

1. Real Numbers

Competency	: 1	Analyses the system of real numbers
Competency Level	:	1.1, 1.2
		Page 3 of the Grade 12 Teacher's Instructional Manual
Subject Content	:	Real Numbers

By studying this section you will develop the skills of

- defining the set of real numbers.
- defining certain subsets of the set of real numbers.
- geometrically representing some real numbers.
- expressing properties of real numbers.
- determining the fraction that is equivalent to a terminating decimal or recurring decimal.
- expressing properties of irrational numbers.
- solving problems involving real numbers.

Introduction

Under this section, sets of numbers will be discussed by first recalling what was learnt by students under the Ordinary level syllabus.

In studying the real number system, the special properties of real numbers will also be discussed. Special attention will be paid to the set of irrational numbers.

1.1 Sets of Numbers

1.1.1 Natural Numbers

$\mathbb{N} = \{1, 2, 3, \ldots\ldots\}$

1.1.2 Integers

$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots.\}$

Positive Integers

$\mathbb{Z}^+ = \{1, 2, 3, \ldots..\}$

Negative Integers

$\mathbb{Z}^- = \{\ldots., -3, -2, -1\}$

Non-negative Integers

$\mathbb{Z}_0^+ = \{0, 1, 2, 3, \ldots..\}$

Non-positive integers

$\mathbb{Z}_0^- = \{\ldots., -3, -2, -1, 0\}$

1.1.3 Rational Numbers

$$\mathbb{Q} = \{x : x = \frac{p}{q}, p, q \in \mathbb{Z}, q \neq 0\}$$

1.1.4 Irrational Numbers

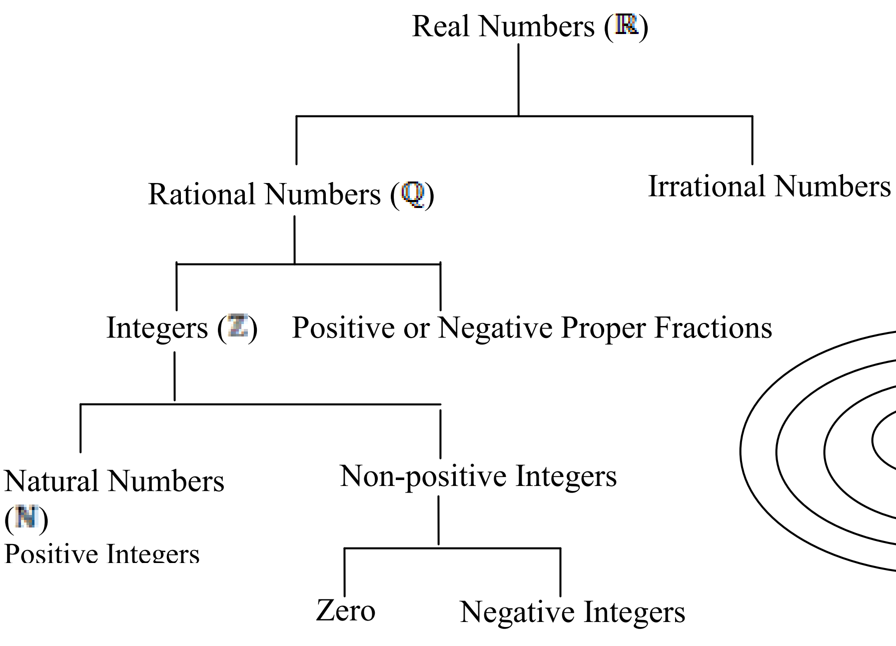
$\sqrt{2}$, $\sqrt{3}$, $\sqrt{11}$, $2\sqrt{2}$, π, e, $(20)^{\frac{1}{4}}$ are examples of irrational numbers. Non-recurring infinite decimal numbers are irrational numbers.

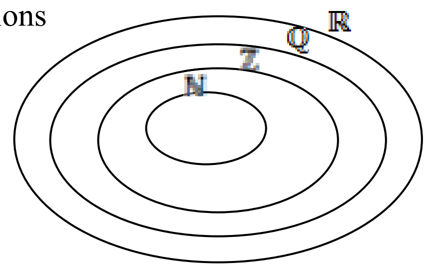
1.1.5 Real Numbers

The set of real numbers can be defined as the union of the set of rational numbers and the set of irrational numbers.
When the set of real numbers is the universal set, it can be written as $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}'$.
Here $\mathbb{Q}'$ is the set of irrational numbers.

$\mathbb{R}^+$ denotes the set of positive real numbers.
$\mathbb{R}^-$ denotes the set of negative real numbers.
$\mathbb{R}_0^+$ denotes the set of non-negative real numbers.
$\mathbb{R}_0^-$ denotes the set of non-positive real numbers.

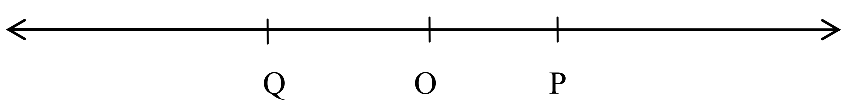
1.2 Certain Subsets of the Set of Real Numbers





1.3 The Geometric Representation of a Real Number

The Number Line (or the Real Line)



Let us mark a fixed point O on a horizontal straight line. If the number zero is represented on the straight line by the point O, each positive real number can be represented uniquely by a point P on the right side of O, and each negative real number can be represented uniquely by a point Q on the left hand side of O.
A straight line such as the above which is used to represent the real numbers is called a number line or a real line. For each point on the real number line there is a corresponding unique real number and to represent each real number, there is a unique point on the number line.

Exercise 1
Present suitable geometric constructions to represent the following irrational numbers on the real number line.

(i) $\sqrt{2}$ (ii) $\sqrt{5}$ (iii) $-\sqrt{7}$ (iv) $2\sqrt{3}$

1.4 Intervals

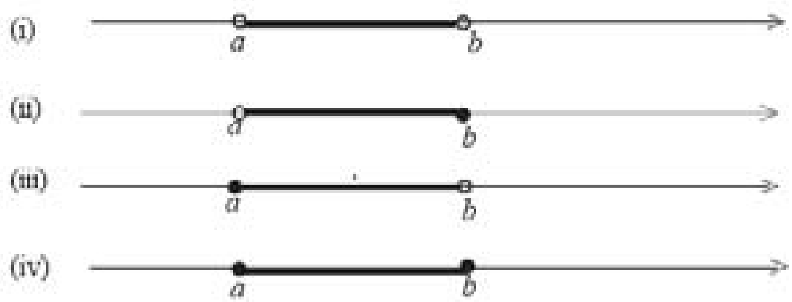
Any segment of a number line represents infinitely many distinct real numbers.

The set of real numbers between any two real numbers a and b (with $a < b$) can be denoted using the following notations

(i)	(a, b)	(when neither a nor b is included in the set)
(ii)	$(a, b]$	(when b is included in the set and a is not included in the set)
(iii)	$[a, b)$	(when a is included in the set and b is not included in the set)
(iv)	$[a, b]$	(when both a and b are included in the set)

These number sets are called intervals.

The above intervals can be represented on a number line in the following manner.



Intervals in set notation

Let $a, b \in \mathbb{R}$.

$(a,b) = \{x : a < x < b\}$

$(a,b] = \{x : a < x \leq b\}$

$[a,b) = \{x : a \leq x < b\}$

$[a,b] = \{x : a \leq x \leq b\}$

a and b are called the end points of the interval.

Note:
(a, b) is called an open interval
(a, b] is called a left open, right closed interval
[a, b) is called a left closed, right open interval
[a, b] is called a closed interval

Exercise 2
Represent the following intervals on a number line.

1. (2, 11) 2. [3, ∞) 3. (-∞, 5] 4. [-8, 7) 5. [-12, 1]

1.5 Properties of Real Numbers

Let a, b, c denote arbitrary real numbers.

1. The sum and the product of two real numbers is a unique real number.
   i.e., $a + b$ and ab denote unique real numbers.

2. Addition and multiplication are commutative.
   i.e., $a+b=b+a$ and
   $ab=ba$.

3. Addition and multiplication are associative.
   i.e., $(a+b)+c=a+(b+c)$.
   Accordingly, this common number can be denoted by $a+b+c$.
   Also, $(ab)c=a(bc)$.
   Accordingly, this common number can be denoted by abc.

4. Multiplication is distributive over addition.
   i.e., $a(b+c)=ab+ac$.

5. (i) An additive identity exists.
   i.e., $a+0=0+a=a$.
   i.e., the result is a, when 0 is added to a or when a is added to 0.

   Therefore, 0 is called the additive identity.
   (ii) A multiplicative identity exists.
   i.e., $a\times 1=1\times a=a$.
   i.e,. the result is a, when a is multiplied by 1 or when 1 is multiplied by a.
   Therefore, 1 is called the multiplicative identity.

6. The existence of inverse elements

   $a+(-a)=(-a)+a=0$.
   i.e., when $-a$ is added to a, or when a is added to $-a$, the additive identity 0 is obtained. Therefore we call $-a$ the additive inverse of a.

   When $a \neq 0$, $a\times\left(\frac{1}{a}\right)=\left(\frac{1}{a}\right)\times a=1$.

i.e., for $a \neq 0$, when a is multiplied by $\left(\frac{1}{a}\right)$, or when $\left(\frac{1}{a}\right)$ is multiplied by a, the multiplicative identity 1 is obtained. Therefore, $\left(\frac{1}{a}\right)$ is called the multiplicative inverse of a.

7. The square of a real number is non-negative (i.e., it is zero or a positive number).

8. Let $n \in \mathbb{Z}^+$ and $a \in \mathbb{R}$.
   (i) If n is even then a^n is non-negative.
   $a^n = 0$ if and only if $a = 0$.
   (ii) If n is odd, then
   a^n is negative when a is negative,
   $a^n = 0$ when $a = 0$, and
   a^n is positive when a is positive.

9. If $n \in \mathbb{Z}^+$, then $\sqrt[n]{a}$ is a real number for n odd. $\sqrt[n]{a}$ is a real number for n even, only if $a \geq 0$.

10. If $n \in \mathbb{Z}^+$ and if neither a nor b is negative, then
$\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$ and $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ (when $b \neq 0$).

1.6 Determining the Fraction equivalent to a Terminating Decimal or Recurring Decimal

Example 1
Determine the fractions equivalent to 0.63 and 0.218.

$$0.63 = \frac{6}{10} + \frac{3}{100} = \frac{60+3}{100} = \frac{63}{100}.$$
$$0.218 = \frac{2}{10} + \frac{1}{100} + \frac{8}{1000} = \frac{200+10+8}{1000} = \frac{218}{1000} = \frac{109}{500}.$$

Example 2
Determine the fractions equivalent to each of the following recurring decimal numbers.
(a) $0.\dot{4}$ (b) $4.\dot{2}\dot{5}$ (c) $2.3\dot{4}\dot{7}$

(a) Method 1
$$\begin{aligned} 0.\dot{4} &= 0.444.... \\ &= 4(0.111....) \\ &= 4\left[\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + ...\right] \end{aligned}$$

$$= \frac{4}{10}\left[1+\frac{1}{10}+\frac{1}{100}+.....\right]$$

$1+\frac{1}{10}+\frac{1}{100}+.....$ is an infinite geometric progression with initial term a equal to 1 and common ratio r equal to $\frac{1}{10}$. Therefore, its sum = $\frac{a}{1-r}=\frac{1}{1-\frac{1}{10}}=\frac{10}{9}$.

$\therefore 0.\dot{4}=\frac{4}{10}\times\frac{10}{9}=\frac{4}{9}$.

Method 2

Let $\quad X = 0.\dot{4}$

Then $\quad X = 0.444...$ --------------------(1)

$\quad 10X = 4.444....$ --------------------(2)

(2) – (1) gives $9X = 4$

Hence $X = \frac{4}{9}$.

(b) Method 1

$4.\dot{2}\dot{5} = 4.252525...$

$= 4 + 0.252525\ldots$

$= 4+\frac{2}{10}+\frac{5}{100}+\frac{2}{1000}+\frac{5}{10\ 000}+\frac{2}{100\ 000}+\frac{5}{1\,000\,000}+....$

$= 4+\frac{25}{100}+\frac{25}{10\,000}+\frac{25}{1\,000\,000}+....$

$= 4+\frac{25}{100}\left[1+\frac{1}{100}+\frac{1}{10\,000}+...\right]$

Here, the progression within brackets is an infinite geometric progression with initial term equal to 1 and common ratio equal to $\frac{1}{100}$.

$$\therefore 4.\dot{2}\dot{5} = 4+\frac{25}{100}\left[\frac{1}{1-\frac{1}{100}}\right]=4+\frac{25}{100}\times\frac{100}{99}=4\frac{25}{99}=\frac{421}{99}.$$

Method 2

Let $\quad X = 4.\dot{2}\dot{5}$

Then $\quad X = 4.252525....$ --------------------(1)

$\quad 100X = 425.252525\ldots.$ --------------------(2)

(2) – (1) gives $99X = 421$.

Hence $X = \frac{421}{99}$.

(a) Method 1

$2.3\dot{4}\dot{7} = 2.3474747...$

$$= 2+\frac{3}{10}+\frac{4}{100}+\frac{7}{1\,000}+\frac{4}{10\,000}+\frac{7}{100\,000}+\frac{4}{1\,000\,000}+\frac{7}{10\,000\,000}+...$$

$$= 2+\frac{3}{10}+\frac{47}{1\,000}+\frac{47}{100\,000}+\frac{47}{10\,000\,000}+....$$

$$= 2\frac{3}{10}+\frac{47}{1\,000}\left[1+\frac{1}{100}+\frac{1}{10\,000}+....\right]$$

Here, the progression within brackets is an infinite geometric progression with initial term equal to 1 and common ratio equal to $\frac{1}{100}$.

Its sum = $\dfrac{1}{1-\dfrac{1}{100}} = \dfrac{100}{99}$

$\therefore 2.3\dot{4}\dot{7} = 2\frac{3}{10}+\frac{47}{1\,000}\times\frac{100}{99} = \frac{23}{10}+\frac{47}{990} = \frac{2324}{990}$.

Method 2

Let $X = 2.3\dot{4}\dot{7}$

Then $X = 2.3474747...$

$10X = 23.474747\ldots$ --------------------(1)

$1000X = 2347.474747\ldots$ --------------------(2)

(2) – (1) gives $990X = 2324$

$\therefore X = \frac{2324}{990}$.

Exercise 3

(1) Determine the fractions equivalent to each of the following decimal numbers.

(i) $0.\dot{2}$ (ii) $3.\dot{1}6\dot{8}$ (iii) $2.\dot{3}$ (iv) $2.\dot{2}\dot{7}$ (v) $0.8\dot{1}\dot{7}$

(2) Determine the fraction equivalent to $\dfrac{3.\dot{6}5\dot{1}\times 0.\dot{6}\dot{8}}{2.\dot{9}6\dot{4}}$.

1.7 Irrational Numbers

When the set of real numbers is the universal set, then the complement of the set of rational numbers is the set of irrational numbers.

Examples: $\sqrt{2}, \sqrt{3}, \sqrt{50}, 17^{\frac{3}{4}}, e, \pi$ are irrational numbers.

[$\pi = 3.141592653689...,\quad e = 2.178281828....$]

Non-recurring decimal numbers are the decimal form of irrational numbers.

Note: $\sqrt{2}$ is only a notation used to represent an irrational number.

1.7.1 Properties of Irrational Numbers

If a is an irrational number and c is a rational number, then

(i) $-a$ is an irrational number.

(ii) $a + c$ is an irrational number.

(iii) ac and $\frac{c}{a}$ are irrational numbers.

If $c \neq 0$, then $\frac{a}{c}$ is an irrational number.

Note:

If a and b are irrational numbers, then $a+b, a-b, ab, \frac{a}{b}, \frac{b}{a}$ could be either rational or irrational.

Exercise 4

Verify each of the instances in the above note using numerical examples.

1.7.2. Rationalization of certain Fractional Expressions involving Surds

- A numerical expression containing square roots, cube roots or n^{th} roots is called a surd.
- $\sqrt{a}-b$ and $\sqrt{a}+b$ are called conjugates of each other.
  $\sqrt{a}-\sqrt{b}$ and $\sqrt{a}+\sqrt{b}$ are also called conjugates of each other.

Example 1

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

Example 2

$$\frac{3}{\sqrt{7}} = \frac{3}{\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}} = \frac{3\sqrt{7}}{7}.$$

Example 3

$$\frac{1}{\sqrt{3}+1} = \frac{1}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1} = \frac{\sqrt{3}-1}{2} = \frac{\sqrt{3}}{2} - \frac{1}{2}.$$

Exercise 5
Rationalize the denominator and simplify.

1. $\frac{1}{\sqrt{3}}$

2. $\frac{2}{\sqrt{7}}$

3. $\frac{2}{\sqrt{32}}$

4. $\frac{1}{2\sqrt{2}-1}$

5. $\frac{2}{3\sqrt{2}+5}$

6. $\frac{2}{\sqrt{6}-\sqrt{5}}$

7. $\frac{1}{(2\sqrt{2}+1)^2}$

2. Functions

Competency : 2 Analyses functions of one variable
Competency Level : 2.1 and 2.2
Page 4 of the Grade 12 Teacher's Instructional Manual
Subject Content : Functions

By studying this section you will develop the skills of

- defining a function.
- defining the domain, co-domain, image, range and natural domain of a function.
- representing a function using function notations.
- expressing the domain and range of a given function.
- defining a constant function, the unit function, the modulus function, a piecewise function and the modulus of a function.
- defining the inverse function.
- writing down the inverse function of a given function, when it exists.
- expressing the conditions for two functions to be equal.
- identifying functions which are equal to each other.
- identifying the special characteristics of the graph of a given function.

Introduction

The concept of a function can be used to explain many phenomena related to the pure sciences (physical, chemical, biological) as well as to predict certain phenomena related to economics and the social sciences.
Several theories in Pure Mathematics too are based on functions.
An accurate definition of "function" was not put forward until many centuries after its concept was first devised by man.
In 1694, Leibniz stated that a function is "a quantity related to a curve".
In 1718, Johann Bernoulli defined a function as "an expression involving constants and variables".
In the same century, Euler defined a function of a variable quantity as "an analytic expression composed in any way from this variable quantity and numbers or constant quantities".

At present, there are many equivalent definitions of a function which are in use. The definition of a function used here is the definition that was first presented by Dirichlet, which is the definition used in the combined mathematics syllabus.

2.1 Definition of a Function

A function f from a non-empty set X to a non-empty set Y is a rule which assigns to each element x in X, a unique element y in Y.

x is called the independent variable, and y is called the dependent variable. The mathematical process which assigns to each element x in X, a unique element y in Y is defined as the 'rule'.

2.2 The Domain, Co-domain, Image, Range and Natural Domain of a Function

2.2.1 Domain (D_f)

If X (non-empty) is the set which consists of exactly the values of the independent variable x of the function f, then X is called the domain of the function f.

2.2.2 Co-domain (C_f)

If all the values that the dependent variable y of a function f takes are elements of a set Y, then Y is called a co-domain of the function f. Accordingly, the co-domain of a function is not unique.

2.2.3 Image

If y is the element assigned by the function f to the element x in the domain X of f, then y is called the image of x under the function f.

2.2.4 Range (R_f)

The set consisting of all the images of the elements in the domain under the function f is called the range of f. The range of a function f is a subset of the co-domain of the function; i.e., $R_f \subseteq C_f$.

2.2.5 Natural Domain

When defining a function, it is necessary to indicate the domain of the function. However, when the domain is not indicated, the set of all real numbers to which a value can be assigned under the function is considered to be its domain. This is defined as the natural domain of the function.

Example:

1. The natural domain of $y = 2x + 3$ is $\mathbb{R}$.

2. $y = \dfrac{1}{x+2}$ is not defined when $x = -2$. $\therefore$ the natural domain of this function is $\mathbb{R}$ - {-2}. This can be written as $(-\infty,-2) \cup (-2,\infty)$ or as $x \in \mathbb{R}$ and $x \neq -2$.

3. $y = \sqrt{4 - x^2}$. Since the square root of a negative number is not a real

number, the natural domain of this function is [-2, 2].

2.2.6 Function Notation

"*y* is a function of *x*" is denoted by $y = f(x)$. Here, *f* represents the rule by which *x* and *y* are related.

For example, if *f* is the rule "add 2 to *x* multiplied by 3", then in function notation it can be represented as

$f(x) = 3x + 2$ or as

$f : x \mapsto 3x + 2$.

Exercise 1

If *x* is a real number, find the natural domain and range of each of the functions given below.

1. $f : x \mapsto 2x - 5$
2. $f : x \mapsto x^2 + 7$
3. $f : x \mapsto \dfrac{1}{x^2 + 2}$
4. $f : x \mapsto x^2 - 6$
5. $f : x \mapsto x^4 - x^2$

2.3 Special Functions

2.3.1 Constant Function

A function *f* is said to be a constant function if there exists *k* such that $f(x) = k$ for all $x \in D_f$.

2.3.2 Unit Function

A constant function defined above such that $k = 1$ is defined as the unit function; i.e., the function *f* such that $f(x) = 1$ for all $x \in D_f$ is called the unit function.

2.3.3 Modulus Function

The modulus function is defined as $f(x) = |x|$ for all $x \in D_f$.

i.e., $f(x) = |x| = \begin{cases} x \ ; & x > 0 \\ 0 \ ; & x = 0 \\ -x \ ; & x < 0 \end{cases}$

2.3.4 Piecewise Function

A piecewise function is a function whose defining rule is different on disjoint intervals of its domain.

Example:

2.3.5 Modulus of a Function

The function F defined by $F(x) = |f(x)|$ for all $x \in D_f$ is called the modulus of the function f. It is denoted by $|f|$.

$$|f|(x) = \begin{cases} -f(x); & f(x) < 0 \\ 0; & f(x) = 0 \\ f(x); & f(x) > 0 \end{cases}$$

Example: Let $f(x) = 2x - 3$.

$$\text{Then } |f|(x) = \begin{cases} -(2x-3); & x < \frac{3}{2} \\ 0; & x = \frac{3}{2} \\ 2x-3; & x > \frac{3}{2} \end{cases}$$

2.3.6 Inverse Function

Let f be a function whose domain is X. If f maps a unique x in X to each y in the range of f, then the inverse of f is the rule which assigns to each y in the range of f, that unique element of X which is mapped by f to y.

Example: Find the inverse function of $f(x) = 4x - 7$.

This is a function with domain $\mathbb{R}$ and range $\mathbb{R}$. If y is the image of x under f, then $y = 4x - 7$. Hence $x = \dfrac{y+7}{4}$; i.e., the correspondence determined by f is reversed by the function given by $g(y)=\dfrac{y+7}{4}$. Since y is a dummy variable, we can re-write this as $g(x)=\dfrac{x+7}{4}$. Thus $f^{-1}(x)=\dfrac{x+7}{4}$.

In some instances, f^{-1} can be defined by restricting the domain of the function f.

Example: Find the inverse function of

Since $x \geq 0$ we reject $x=-\sqrt{36-y}$.

$$\therefore x=\sqrt{36-y}, \quad g(y)=\sqrt{36-y}$$
$$\therefore f^{-1}(x)=\sqrt{36-x}; \quad 0 \leq x \leq 36$$

The domain of f is $\{x: 0 \leq x \leq 6\}$, the range of f is $\{y: 0 \leq y \leq 36\}$
The domain of f^{-1} is $\{x: 0 \leq x \leq 36\}$, the range of f^{-1} is $\{y: 0 \leq y \leq 6\}$

2.4 Equality of Two Functions

Let us consider two function f and g. If the following two conditions are satisfied, then f and g are said to be equal.

(i) $D_f = D_g$.
(ii) For all $x \in D_f$ (or $x \in D_g$), $f(x)=g(x)$.

This is represented by $f = g$. In this situation, it is clear that $R_f = R_g$.

Example:

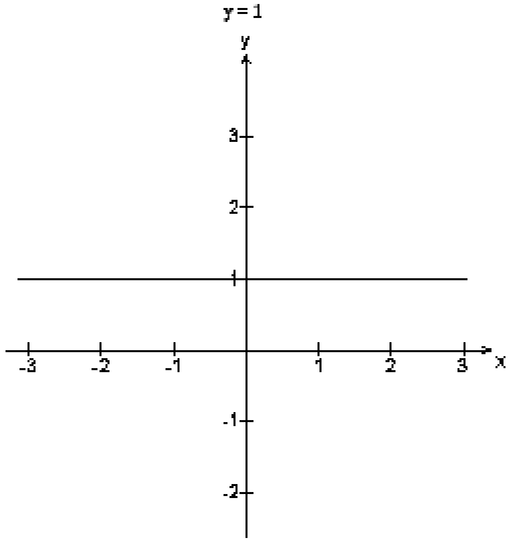
1. If $f(x)=\dfrac{1}{x}-x$ and $g(x)=\dfrac{1-x^2}{x}$, then $f = g$. This is because the natural domains of both f and g are both $\mathbb{R}$ -{0} and $f(x)=g(x)$ for all $x \in \mathbb{R}$ -{0}.
2. $f(x)=\sin x, \quad g(x)=\cos\left(\dfrac{\pi}{2}-x\right)$. Then $f = g$. The common natural domain is $\mathbb{R}$.

2.5 Graph of a Function

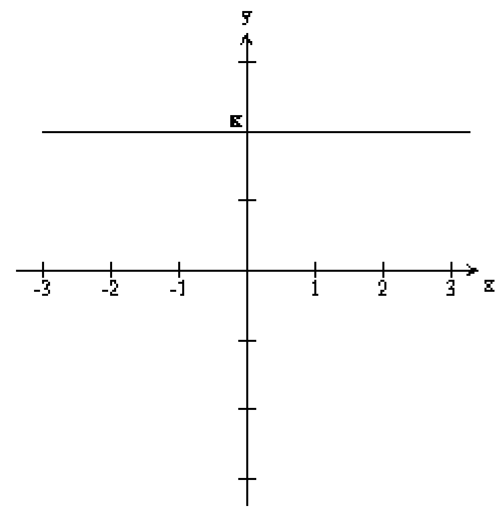
Let f be a given function. The graph of f is the set $\{(x, f(x)) : x \in D_f\}$. The graph of a real function can be represented on a Cartesian plane.

2.5.1 Examples of Graphs of Special Types of Functions

(a) The graph of the unit function



(b) The graph of a constant function

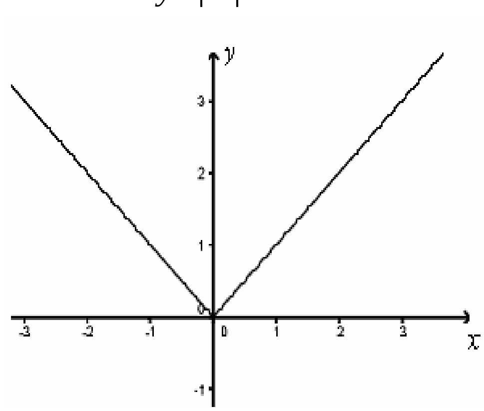


c) The graph of $f(x) = |x|$

$$f(x) = |x| = \begin{cases} x \ ; \ x > 0 \\ 0 \ ; \ x = 0 \\ -x \ ; \ x < 0 \end{cases}$$

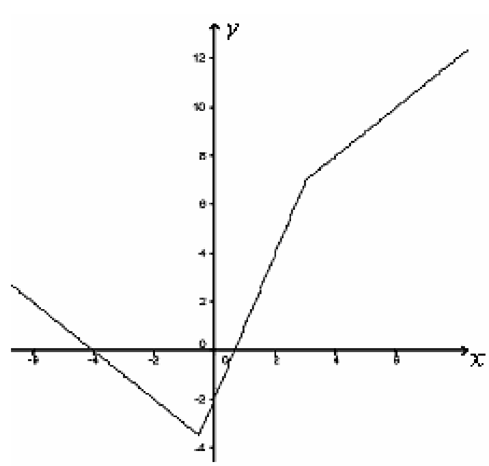
By keeping that portion of the graph of $y = x$ which is above the x-axis as it is, and reflecting the portion of the graph of $y = x$ which is below the x-axis over the x – axis, the graph of $y = |x|$ can be obtained.

$y = |x|$



(d) The graph of

$$f(x) = \begin{cases} -x-4 \ ; \ x \leq -\frac{1}{2} \\ 3x-2 \ ; \ -\frac{1}{2} < x \leq 3 \\ x+4 \ ; \ x > 3 \end{cases}$$

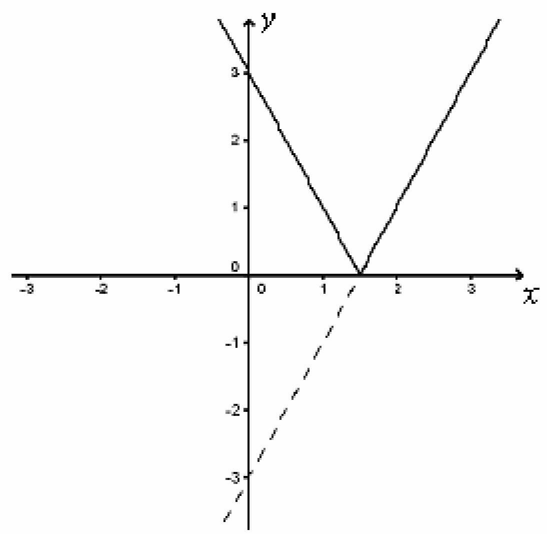


(e) The graph of the modulus of a function

$y = |2x - 3|$

$$y = |2x-3| = \begin{cases} -(2x-3)\ ; & x < \frac{3}{2} \\ 0\ ;\ x = \frac{3}{2} \\ 2x-3\ ;\ x > \frac{3}{2} \end{cases}$$

$$= \begin{cases} -2x+3\ ; & x < \frac{3}{2} \\ 0\ ;\ x = \frac{3}{2} \\ 2x-3\ ;\ x > \frac{3}{2} \end{cases}$$



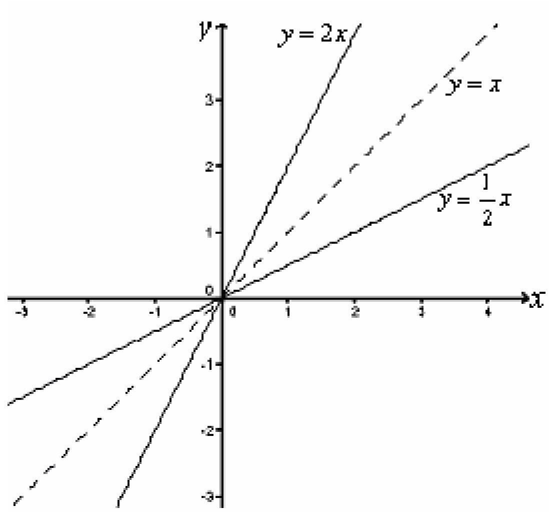
By taking *y* as a piecewise function as above and plotting the graph, or by plotting the graph of $y = 2x - 3$ and reflecting that portion of the graph which is below the *x*-axis over the *x*-axis, the graph of $y = |2x - 3|$ can be obtained as shown in the figure.

2.5.2 The Graph of the Inverse of a Function

When the inverse f^{-1} of the function *f* exists, if $f(x) = y$ then $f^{-1}(y) = x$.
$\therefore$ (x, y) is a point on the graph of *f* if and only if (y, x) is a point on the graph of f^{-1}. It is clear from this that the graphs of *f* and f^{-1} are reflections of each other over the straight line $y = x$.

Example:

(a) Let us consider the function $f(x) = 2x$.



(b) Let us consider the function $f(x) = x^2; \quad x \in \mathbb{R}_0^+$.

$y = x^2$

$\Rightarrow x = \pm\sqrt{y}$.

Since x is non negative,

we reject $x = -\sqrt{y}$,

and hence $x = \sqrt{y}$.

Exercise 2

(1) Determine the range of each of the following functions.

(i) $f(x) = x + 4; \quad 0 < x < 5$.

(ii) $f : x \to (x^2 + 3)^2; \quad x \in \mathbb{R}$.

(iii) $f : x \to 5x^3 - 1; \quad 1 < x < 3$.

(iv) $f : x \to 3\sqrt{x} - 4; \quad x \in \mathbb{R}^+$.

(v) $f : x \to \sqrt{3x - 2}; \quad 2 \le x \le 9$.

(vi) $f : x \to \dfrac{1}{3 + x^4}; \quad x \in \mathbb{R}$.

(2) Determine the range of each of the following functions by drawing the graph.

(i) $f(x) = \begin{cases} 3x + 4; & 0 \le x \le 4 \\ x \quad ; & 4 < x \le 6 \end{cases}$

(ii) $g(x) = \begin{cases} x^2 \quad ; & 0 \le x \le 3 \\ 12 - x; & 3 < x \le 12 \end{cases}$

(iii) $h(x) = \begin{cases} (x + 2)^2; & -1 \le x \le 0 \\ 4 \quad ; & 0 < x \le 3 \\ 7 - x \quad ; & 3 < x \le 6 \end{cases}$

(iv) $f(x)=\begin{cases} x^3 & ; \quad 0 \le x \le 2 \\ 2x+4 & ; \quad 2 < x \le 6 \\ 16-(x-6)^2 & ; \quad 6 < x \le 10 \end{cases}$

(3) Determine the inverse function of each of the following functions.

$(i)\ f(x) = 3x + 2$

$(ii)\ f(x) = \dfrac{3}{x-1}\ ; x \neq 1$

$(iii)\ f(x) = 1 + \dfrac{1}{x}\ ; x \neq 0$

$(iv)\ f(x) = 2 - \dfrac{3}{4+x}\ ; x \neq -4$

(4) Determine the inverse function of each of the following functions, and the range of each inverse function.

(i) $f(x) = x^2; \quad x > 2.$

(ii) $f(x) = \dfrac{1}{2+x}; \quad x > 0.$

(iii) $f(x) = \sqrt{x-2}; \quad x > 3.$

(iv) $f(x) = \dfrac{1}{x} - 3; \quad 2 < x < 5.$

(v) $f(x) = 5 - \sqrt{x+3}; \quad x \geq -3.$

(5) The domain of $h(x) = \dfrac{4}{x+3}$ is $\{x : x \geq 0\}$.

(i) Draw the graph of h and determine its range.

(ii) Find $h^{-1}(x)$.

(iii) Determine the values of x for which $h(x) = h^{-1}(x)$.

3. Polynomial Functions and Rational Functions

Competency	: 4	Manipulates Polynomial Functions
	5	Resolves Rational Functions into Partial Fractions
Competency Level	:	4.1, 4.2, 4.3 and 5.0 Pages 10, 11, 13, 14 of the Grade 12 Teacher's Instructional Manual
Subject Content	:	Algebra of Polynomial Functions Rational Functions

By studying this section you will develop the skills of

- identifying a polynomial of one variable.
- identifying linear functions, quadratic functions and cubic functions.
- identifying the conditions under which two polynomials are identical.
- understanding the basic operations on polynomials.
- dividing a polynomial by another polynomial.
- expressing the division algorithm.
- expressing the remainder theorem.
- proving the remainder theorem.
- expressing the factor theorem.
- proving the factor theorem.
- expressing the converse of the factor theorem.
- proving the converse of the factor theorem.
- solving problems using the remainder theorem and the factor theorem.
- solving certain polynomial equations.
- defining the zeros of a polynomial.
- defining rational expressions.
- defining rational functions.
- defining proper rational expressions and improper rational expressions.
- resolving rational expressions into partial fractions.

Introduction

This section deals with the subject content on polynomials and rational functions under the combined mathematics syllabus. It presents solved problems in a manner suitable for the students, so that they will develop the competency of solving from basic concepts, simple problems and then gradually more advanced problems. By this, both students' and teachers' tasks are facilitated and the learning-teaching process can be improved.

3.1 Polynomials in one variable

When *n* is a non-negative integer, an expression of the form $a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$ where $a_n \neq 0$ is called a polynomial. Here, $a_n, a_{n-1}, a_{n-2}, ...a_1, a_0$ are real constants. The general term is $a_r x^r$. r is called the degree of *x* in the term $a_r x^r$. It is also the degree of the term $a_r x^r$. The term in a polynomial with the highest degree is called the leading term and its coefficient is called the leading coefficient. The degree of the polynomial is the degree of its leading term. Accordingly, the degree of the above polynomial is *n*.

The above polynomial has a unique value corresponding to each value of *x*. Therefore a polynomial is expressed as a function of *x*. Thus in function notation, it can be written as

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_2 x^2 + a_1 x + a_0$$

Example:

1. $f(x) = 3$ is a constant polynomial. Its degree is 0.

2. $f(x) = 2x - 1$ is a linear polynomial of degree 1.

3. $f(x) = 3x^2 - 4x + 2$ is a quadratic polynomial of degree 2.

4. $f(x) = 5x^4 - 5x^3 + 6x + 4$ is a polynomial of degree 4 with leading coefficient equal to 5.

   Here the coefficient of x^2 is zero.

3.2 Identical Polynomials

Two polynomials are said to be identical if their degrees are the same and if the coefficients corresponding to the terms of equal degree in the two polynomials are the same.

Example: Let $f(x) = 3x^2 - 2x - 5$ and $g(x) = (3x-5)(x+1)$.

Then $f \equiv g$.

The symbol '≡' is used to represent 'identical'.

3.3. Basic Mathematical Operations on Polynomials.

The basic mathematical operations addition, subtraction, multiplication and division can be performed on polynomials as shown in the examples below.

If *f* and *g* are two polynomials, then

1. $f + g$ is given by $(f + g)(x) = f(x) + g(x)$
2. $f - g$ is given by $(f - g)(x) = f(x) - g(x)$
3. fg is given by $(fg)(x) = f(x)g(x)$
4. $\frac{f}{g}$; $g(x) \neq 0$, is given by $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

Example: If $f(x) = x^2 + 4x - 8$ and $g(x) = x + 2$, then

1. $f(x) + g(x) = x^2 + 4x - 8 + x + 2$

   $f(x) + g(x) = x^2 + 5x - 6$

2. $f(x) - g(x) = x^2 + 4x - 8 - x - 2$

   $f(x) - g(x) = x^2 + 3x - 10$

3. $f(x)g(x) = (x^2 + 4x - 8)(x + 2)$

4. $\frac{f(x)}{g(x)} = \frac{x^2 + 4x - 8}{x + 2}$

3.4 Dividing a polynomial by another polynomial

When a polynomial $f(x)$ is divided by a polynomial $g(x)$, if the quotient is $k(x)$, and the remainder is $R(x)$, we can write $f(x) = k(x)g(x) + R(x)$; this is called the division algorithm.

$f(x)$ is called the dividend and $g(x)$ is called the divisor.

Two methods are adopted when dividing one polynomial by another.

(i) Long Division

(ii) Synthetic Division

Example 1: Let us divide the polynomial $4x^3 - 2x^2 + 5x + 3$ by the linear polynomial $(x - 2)$.

(i) By the long division method

$$\begin{array}{r|l} & 4x^2+6x+17 \\ \hline x-2 & 4x^3-2x^2+5x+3 \\ & 4x^3-8x^2 \\ \hline & \quad 6x^2+5x+3 \\ & \quad 6x^2-12x \\ \hline & \qquad 17x+3 \\ & \qquad 17x-34 \\ \hline & \qquad\qquad 37 \end{array}$$

Here the quotient is : $4x^2 + 6x + 17$

The remainder is : 37

(ii) By the method of synthetic division

$$\begin{array}{r|rrrr} 2 & 4 & -2 & 5 & 3 \\ \hline & & 8 & 12 & 34 \\ & 4 & 6 & 17 & 37 \\ \hline \end{array}$$

Here 4, 6 and 17 are the coefficients of the quotient $4x^2 + 6x + 17$, and the remainder is 37.

Example 2: Let us divide the polynomial $3x^2 + 4x - 1$ by x + 3.

(i) By the method of long division

$$
\begin{array}{r|l}
 & 3x^2 \quad - 9x + 31 \\
\hline
x + 3 & 3x^3 \quad + 4x - 1 \\
 & 3x^3 + 9x^2 \\
\hline
 & \quad - 9x^2 + 4x \\
 & \quad -9x^2 - 27x \\
\cline{2-2}
 & \qquad\quad 31x - 1 \\
 & \qquad\quad 31x + 93 \\
\cline{2-2}
 & \qquad\qquad -94
\end{array}
$$

(ii) By the method of synthetic division

$$
\begin{array}{r|rrrr}
-3 & 3 & 0 & 4 & -1 \\
\cline{2-5}
 & & -9 & 27 & -93 \\
 & 3 & -9 & 31 & 94 \\
\cline{2-5}
\end{array}
$$

Here 3, -9 and 31 are the coefficients of the quotient $3x^2 - 9x + 31$ and the remainder is -94.

3.5 The Remainder Theorem

When the polynomial $f(x)$ is divided by $(x - a)$, the remainder is $f(a)$.

Proof:

Let $\varphi(x)$ be the quotient and R be the remainder when the polynomial $f(x)$ is divided by $(x - a)$.

Then $f(x) \equiv (x - a)\varphi(x) + R$, by the division algorithm.

By substituting $x = a$ we obtain, $f(a) = (a - a)\varphi(a) + R$.

Therefore, $f(a) = R$

Application of the Remainder Theorem:

Example : Let us find the remainder when $2x^3 - 3x^2 + 2x - 1$ is divided by $(x - 2)$.

Let us consider $f(x) = 2x^3 - 3x^2 + 2x - 1$.

By substituting $x = 2$ we obtain

$f(2) = 2 \times 2^3 - 3 \times 2^2 + 2 \times 2 - 1$

$f(2) = 16 - 12 + 4 - 1 = 7$

Therefore the remainder is 7.

3.6 The Factor Theorem

If $f(a) = 0$, then $(x - a)$ is a factor of the polynomial $f(x)$. (This is a special instance of the remainder theorem)

Proof:

Since $f(a) = 0$, when the polynomial $f(x)$ is divided by $(x - a)$, the remainder is 0. Therefore, $(x - a)$ is a factor of $f(x)$.

The converse of the factor theorem

If $(x - a)$ is a factor of the polynomial $f(x)$, then $f(a) = 0$.

Proof:

Since $(x - a)$ is a factor of the polynomial $f(x)$, it can be expressed as $f(x) \equiv (x - a)\varphi(x)$.

When we substitute $x = a$ into this we obtain, $f(a) = (a - a)\varphi(a) = 0$

Example 1: Finding the factors of $x^3 - x^2 - x + 1$.

Method 1: Let $f(x) = x^3 - x^2 - x + 1$.

When $x = 1$, $f(1) = 1^3 - 1^2 - 1 + 1 = 0$.

$\therefore$ $(x - 1)$ is a factor of $f(x)$.

When $x = -1$, $f(-1) = (-1)^3 - (-1)^2 - (-1) + 1 = -1 - 1 + 1 + 1 = 0$.

$\therefore$ $(x + 1)$ is a factor of $f(x)$.

$(x-1)(x+1)$ is a polynomial of degree two. Since f is a polynomial of degree 3, the remaining factor should be a linear factor in x. Suppose it is $ax+b$.

Then $f(x)=(x-1)(x+1)(ax+b)$

$x^3-x^2-x+1=(x-1)(x+1)(ax+b)$

By comparing the coefficient of x^3 we obtain; 1 = a.

By comparing the constant coefficient we obtain; -1 = b.

$\therefore\ f(x)=(x-1)(x+1)(x-1)$

$f(x)=(x-1)^2(x+1)$

Method 2: Apart from the above method, the factors of a polynomial can also be found by first applying the factor theorem as in the first method to obtain a linear factor, and then, to find the other factors, dividing the polynomial by this factor, and factoring in the usual method, the quotient that is obtained.

i.e., Let $f(x)=x^3-x^2-x+1$.

When x = 1, $f(1)=1^3-1^2-1+1=0$. $\therefore$ $(x-1)$ is a factor of $f(x)$.

$$
\begin{array}{r|l}
 & x^2 \quad\quad - 1 \\
\cline{2-2}
x - 1 & x^3 - x^2 - x + 1 \\
 & x^3 - x^2 \\
\cline{2-2}
 & \quad\quad\quad -x + 1 \\
 & \quad\quad\quad -x + 1 \\
\cline{2-2}
 & \quad\quad\quad\quad\quad 0
\end{array}
$$

Factoring the quotient,

$f(x)=(x-1)(x^2-1)$

$=(x-1)(x+1)(x-1)$

$=(x-1)^2(x+1)$

Example 2: x^2-4 is a factor of the cubic expression $x^3+cx^2+dx-12$. Find the values of c and d and thereby determine the other factors.

Let $f(x)=x^3+cx^2+dx-12$

$x^2-4=(x-2)(x+2)$

Since this is a factor of $f(x)$, when x = 2,

$f(2)=2^3+c\times 4+2\times d-12=0$

$\therefore \quad 4c+2d-4=0$ ----------------------- (1)

When x = - 2,

$f(-2)=(-2)^3+c\times(-2)^2+(-2)\times d-12=0$

$\therefore \quad 4c-2d-20=0$ ----------------------- (2)

By (1) + (2), 8c = 24. $\therefore c=3$.

By (1) – (2), 4d = -16. $\therefore d=-4$.

$\therefore f(x)=x^3+3x^2-4x-12$.

Since f is a polynomial of degree 3, the other factor should be linear. Let us take it as $ax+b$. Then,

$(x^2-4)(ax+b)\equiv x^3+3x^2-4x-12$

By considering the coefficient of x^3; a = 1

By considering the constant coefficient; -4b = -12. $\therefore b=3$.

$\therefore$ the factors of $f(x)$ are $(x-2),(x+2)$ and $(x+3)$.

3.7 Rational Expressions

If $P(x)$ and $Q(x)$ are polynomials, an expression of the form $\dfrac{P(x)}{Q(x)}, Q(x)\neq 0$ is called a rational expression.

3.7.1. Proper Rational Functions

If the degree of $P(x)$ < the degree of $Q(x)$, then the function $f(x) = \frac{P(x)}{Q(x)}$ is called a proper rational function, or a proper fraction.

3.7.2. Improper Rational Functions

If the degree of $P(x) \geq$ the degree of $Q(x)$, then the function $f(x) = \frac{P(x)}{Q(x)}$ is called an improper rational function, or an improper fraction.

Example (1) $f(x) = \dfrac{7x+8}{(x+4)(x-6)}, \quad x \neq -4, 6$

(2) $f(x) = \dfrac{9x^2+34x+14}{(x+2)(x^2-x-12)}, \quad x \neq -2,\ 4,\ -3$

(3) $f(x) = \dfrac{5x^2-71}{(x+5)(x-4)}, \quad x \neq -5,\ +4$

(4) $f(x) = \dfrac{x^3+3x^2-2x+1}{(x-1)(x-2)}, \quad x \neq 1,\ 2$

(1) and (2) above are examples of proper rational functions while (3) and (4) are examples of improper rational functions.

3.7.3. Resolving a Proper Rational Function into Partial Fractions

(a) When the denominator consists of only linear factors.

Example: $\dfrac{2x+5}{(x+1)(x+2)} \equiv \dfrac{\mathrm{A}}{x+1} + \dfrac{\mathrm{B}}{x+2}$

Multiplying both sides by $(x+1)(x+2)$ we obtain

$2x + 5 \equiv \mathrm{A}(x + 2) + \mathrm{B}(x + 1)$

When $x = -1$; $3 = \mathrm{A}(-1 + 2)$. $\therefore 3 = \mathrm{A}$

When $x = -2$; $-4 + 5 = \mathrm{B}(-2 + 1)$. $\therefore -1 = \mathrm{B}$

$$\frac{2x+5}{(x+1)(x+2)} \equiv \frac{3}{(x+1)} + \frac{-1}{(x+2)}$$

(b) When the denominator has quadratic factors

Example: $f(x) = \dfrac{2-6x+10x^2}{(1-3x)(1+x^2)}$.

$$\frac{2-6x+10x^2}{(1-3x)(1+x^2)} \equiv \frac{A}{1-3x} + \frac{Bx+C}{1+x^2}$$

$$\therefore 2-6x+10x^2 = A(1+x^2)+(Bx+C)(1-3x)$$

Comparing the coefficient of x^2 on the two sides; 10 = A – 3B ---------- (1)

Substituting $x = \frac{1}{3}$;

$$2-6\times\frac{1}{3}+10\left(\frac{1}{3}\right)^2 = A\left[1+\left(\frac{1}{3}\right)^2\right]$$

$$\frac{10}{9} = A\left(\frac{10}{9}\right);\ \ 1 = A$$

Substituting A = 1 in (1);

10 = 1 -3B

$$\frac{9}{-3} = B$$

$\therefore$ B = -3.

Comparing the constant coefficient on the two sides;

2 = A + C.

Substituting A = 1 we obtain C = 1.

Hence $\dfrac{2-6x+10x^2}{(1-3x)(1+x^2)} \equiv \dfrac{1}{1-3x} + \dfrac{-3x+1}{1+x^2}$

(c) When the denominator has a repeated quadratic factor

Example: $f(x) = \dfrac{2x^3+3x^2-5x+1}{(x^2-2x+3)^2}$

$$\frac{2x^3+3x^2-5x+1}{(x^2-2x+3)^2} \equiv \frac{\mathrm{A}x+\mathrm{B}}{x^2-2x+3} + \frac{\mathrm{C}x+\mathrm{D}}{(x^2-2x+3)^2}$$

$$2x^3+3x^2-5x+1 \equiv (\mathrm{A}x+\mathrm{B})(x^2-2x+3)+\mathrm{C}x+\mathrm{D}$$

Comparing the coefficient of x^3; A = 2

Comparing the coefficient of x^2; 3 = -2A + B. Then B = 7.

Comparing the coefficient of x ; -5 = 3A – 2B + C. Then C = 3.

Comparing the constant coefficient; 1 = 3B + D

1 = 21 + D

D = -20

$$\therefore \frac{2x^3+3x^2-5x+1}{(x^2-2x+3)^2} \equiv \frac{2x+7}{(x^2-2x+3)} + \frac{3x-20}{(x^2-2x+3)^2}$$

3.7.4 Resolving Improper Rational Functions into Partial Fractions

Example: $f(x) = \dfrac{5x^2-71}{(x+5)(x-4)}$

$$\frac{5x^2-71}{(x+5)(x-4)} \equiv \mathrm{A} + \frac{\mathrm{B}}{(x+5)} + \frac{\mathrm{C}}{(x-4)}$$

$$5x^2-71 \equiv \mathrm{A}(x+5)(x-4)+\mathrm{B}(x-4)+\mathrm{C}(x+5)$$

When x = -5; $5(-5)^2-71 = \mathrm{B}(-5-4)$

54 = -9B. $\therefore$ -6 = B

When x = 4; $5(4)^2-71 = C(4+5)$

9 = 9C. $\therefore$ 1 = C

Comparing the coefficient of x^2 on the two sides; 5 = A.

$$\therefore \frac{5x^2-71}{(x+5)(x-4)} \equiv 5 - \frac{6}{(x+5)} + \frac{1}{(x-4)}$$

Example:

$f(x) = \dfrac{3x^4+7x^3+8x^2+53x-186}{(x+4)(x^2+9)}$

$$\frac{3x^4+7x^3+8x^2+53x-186}{(x+4)(x^2+9)} = \mathrm{A}x+\mathrm{B}+\frac{\mathrm{C}}{(x+4)}+\frac{\mathrm{D}x+\mathrm{E}}{x^2+9}$$

Multiplying both sides by $(x+4)(x^2+9)$;

$$3x^4+7x^3+8x^2+53x-186 \equiv (\mathrm{A}x+\mathrm{B})(x+4)(x^2+9)+\mathrm{C}(x^2+9)+(\mathrm{D}x+\mathrm{E})(x+4)$$

Comparing the coefficient of x^4 on the two sides; 3 = A

Comparing the coefficient of x^3 on the two sides; 7 = B + 4A

B = 7 - 12 = -5

When x = - 4; $3(-4)^4+7(-4)^3+8(-4)^2+53(-4)-186 = \mathrm{C}[(-4)^2+9]$

50 = 25C

2 = C

Comparing the coefficient of x^2 ; 8 = 9A + 4B + C + D.

By substituting A = 3, B = -5 and C = 2 we obtain

8 = 27 – 20 + 2 + D

$\therefore$ D = -1.

Comparing the constant coefficient on the two sides; -186 = 36B + 9C + 4E.

Substituting B = -5 and C = 2 we obtain E = -6.

$$\therefore \frac{3x^4+7x^3+8x^2+53x-186}{(x+4)(x^2+9)} \equiv 3x-5+\frac{2}{(x+4)}+\frac{-x-6}{(x^2+9)}$$

$$\equiv 3x-5+\frac{2}{(x+4)}-\frac{x+6}{x^2+9}$$

Exercises

1. Factorize $a^2(b-c)+b^2(c-a)+c^2(a-b)$.

2. Show that $a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+ab+bc+ca)$

   Hence deduce that $(x-y)^3+(y-z)^3+(z-x)^3=3(x-y)(y-z)(z-x)$

3. i. Find the remainder when $2x^4+5x^2+6$ is divided by $(x+5)$ and thereby express the division algorithm.

   ii. Apply the method of synthetic division to divide $2x^4+5x^2+6$ by $(x+5)$.

4. The remainder when the polynomial $f(x)=ax^3+bx^2+3x-1$ is divided by $(x-1)$ is 2. When it is divided by $(x+1)$ the remainder is 3. Determine a and b. Find the remainder when the polynomial $f(x)$ is divided by (x^2-1).

5. The polynomial $f(x)=x^8+2x^7+ax^2+bx+c$ is divisible by x^2+x-2 without remainder. If the remainder is -8 when it is divided by $(x+1)$, determine a, b and c.

6. Show that the polynomial $f(x)=4x^4-12x^3+25x^2-24x+16$ is the perfect square of a polynomial of degree 2.

7. If x^3+ax+b and ax^3+bx^2+x-a have a common linear factor, show that this common linear factor is a factor of $(b-a^2)x^2+x-a(1+b)$ too.

8. Resolve the following into partial fractions

   i. $\dfrac{x-1}{(x+1)(x+3)}$

   ii. $\dfrac{x}{(x-3)(x^2+x+1)}$

   iii. $\dfrac{2+x^2}{(2-x)^2(x+4)}$

   iv. $\dfrac{3x^2+4x+7}{x^2-3x+2}$

   v. $\dfrac{1+x}{(1-x)(1+x^2)}$

   vi. $\dfrac{9}{(1-2x)(1+x)^2}$

   vii. $\dfrac{x^2}{(x^2+1)(x^2+2)}$

   viii. $\dfrac{2x+1}{(x-1)(x^3+x^2+x+2)}$

   ix. $\dfrac{x^3}{x^2-3x+2}$

   x. $\dfrac{x^4-5x^3+7}{x^3-8}$

4. Quadratic Functions and Quadratic Equations

Competency 3 : Analyses quadratic functions
Competency Level : 3.1, 3.2
Page 3 of the Grade 12 Teacher's Instructional Manual
Subject Content : Quadratic Functions and Quadratic Equations

By studying this section you will develop the skills of

- identifying as quadratic functions, functions of the form $f(x) = ax^2 + bx + c$ where $a(\neq 0), b, c \in \mathbb{R}$
- describing the properties of quadratic functions.
- explaining the various instances of the graph of a quadratic function.
- recognizing that if $f(x)$ is a quadratic function, then $f(x) = 0$ is a quadratic equation.
- expressing the nature of the roots of the quadratic equation $ax^2 + bx + c = 0$ when
  (i) $b^2 - 4ac > 0$ (ii) $b^2 - 4ac = 0$ (iii) $b^2 - 4ac < 0$.
- expressing the sum and product of the roots of a quadratic equation in terms of the coefficients of the quadratic expression.
- constructing quadratic equations whose roots are symmetric functions of the roots of a given quadratic equation.
- solving problems involving quadratic functions and quadratic equations.

Introduction

The concept of the quadratic equation which arose as a result of trying to determine the solution to certain mathematical problems in various fields such as the social sciences, economics and the pure sciences, pre-dates even the concept of zero.
Not only quadratic equations, but cubic equations too were solved before the concept of zero was developed by man. (But it was not as easy a task as is now with the use of zero).

As a whole, the concept of a function developed through various algebraic equations and expressions. The concept of a quadratic function developed through quadratic equations.

Despite this historic development it is more productive to first study quadratic functions, and then study quadratic equations as quadratic functions taking the value zero. Therefore, this lesson has been presented in that order.

4.1 Quadratic Functions

The general form of a quadratic function is $f(x) = ax^2 + bx + c$, where $a \neq 0$.
By completing the squares of $ax^2 + bx + c$, we obtain

$$ax^2 + bx + c = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right] .$$

$b^2 - 4ac$ is called the discriminant of $ax^2 + bx + c$ and is denoted by Δ.

Then $ax^2 + bx + c = a\left[\left(x + \dfrac{b}{2a}\right)^2 - \dfrac{\Delta}{4a^2}\right]$.

If $\Delta > 0$, the quadratic expression has two real factors.
If $\Delta = 0$, the quadratic expression can be written as a perfect square.
If $\Delta < 0$, the quadratic expression cannot be resolved into real factors.

Example (i)

$3x^2 - 2x - 1$

$\Delta = (-2)^2 - 4\times 3\times(-1) = 4 + 12 = 16 > 0$

Therefore the given expression can be resolved into two real factors.

Also, $\Delta = 16 = 4^2$. This is a perfect square. Therefore, the two factors are rational.

$3x^2 - 2x - 1 = (3x+1)(x-1)$

Example (ii)

$2x^2 + 2x - 1$

$\Delta = 2^2 - 4\times 2\times(-1) = 4 + 8 = 12$

Here, although $\Delta > 0$, it is not a perfect square. Thus the expression can be resolved into two distinct real factors.

$$\begin{aligned}
2x^2 + 2x - 1 &= 2\left[x^2 + x - \frac{1}{2}\right] = 2\left[\left(x + \frac{1}{2}\right)^2 - \frac{1}{2} - \frac{1}{4}\right] = 2\left[\left(x + \frac{1}{2}\right)^2 - \frac{3}{4}\right] \\
&= 2\left[\left(x + \frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2\right] = 2\left(x + \frac{1}{2} + \frac{\sqrt{3}}{2}\right)\left(x + \frac{1}{2} - \frac{\sqrt{3}}{2}\right) \\
&= 2\left[x + \left(\frac{1+\sqrt{3}}{2}\right)\right]\left[x + \left(\frac{1-\sqrt{3}}{2}\right)\right]
\end{aligned}$$

Here, although the given expression can be resolved into two factors of the form $a(x-\alpha)(x-\beta)$, α and β are irrational.

Example (iii)

$4x^2 + 4x + 1$

$\Delta \;=\; 4^2 - 4\times 4\times 1 \;=\; 16 - 16 \;=\; 0$

Here $\Delta = 0$.

Also,

$4x^2 + 4x + 1 = (2x+1)^2$

Example (iv)

$3x^2 - x + 2$

$\Delta \;=\; (-1)^2 - 4\times 3\times 2 \;=\; 1 - 24 \;=\; -23 \;<\; 0$

Here $\Delta < 0$.

Therefore, $3x^2 - x + 2$ cannot be resolved into two real factors.

4.1.1 The Sign of Quadratic Functions

$$ax^2 + bx + c \;=\; a\left[\left(x + \frac{b}{2a}\right)^2 \;-\; \frac{\Delta}{4a^2}\right]$$

When $\Delta < 0$, the sign of the function $f(x) = ax^2 + bx + c$ is the same as the sign of *a*.

$\therefore$ when $\Delta < 0$, the sign of the function $f(x) = ax^2 + bx + c$ is the same for all values of *x*.

When $\Delta > 0$, $ax^2 + bx + c$ can be factored into a product of two real factors as

$ax^2 + bx + c \;=\; a(x-\alpha)(x-\beta)$.

When $\alpha < \beta$, to examine the sign of $ax^2 + bx + c$, the intervals $x < \alpha$, $\alpha < x < \beta$ and $x > \beta$ should be considered. A table such as the following may be used for this.

	$x < \alpha$	$\alpha < x < \beta$	$x > \beta$
$(x-\alpha)$	-	+	+
$(x-\beta)$	-	-	+
$(x-\alpha)(x-\beta)$	+	-	+
$a(x-\alpha)(x-\beta)$	the same sign as *a*	the sign opposite to that of *a*	the same sign as *a*

Therefore, in this case, the function takes positive values in some intervals of *x*, and negative values in other intervals; i.e., the sign of the function is different in different intervals of *x*.

Before discussing about the sign of a quadratic function, first determine the sign of the discriminant Δ.

If $\Delta < 0$, complete the square and write the function as a sum of two squares.

If $\Delta = 0$, write the function as a perfect square.
If $\Delta > 0$, resolve the function into a product of rational or irrational factors.

Example (i) $f(x) = x^2 + x + 2$. Here, $\Delta = 1 - 8 < 0$. Therefore, let us complete the square.

$$x^2 + x + 2 \;=\; \left(x + \frac{1}{2}\right)^2 + 2 - \frac{1}{4} \;=\; \left(x + \frac{1}{2}\right)^2 \;+\; \frac{7}{4} \;>\; 0$$

Thus the function is positive for all values of *x*.

Example (ii) $f(x) = -2x^2 + x - 1$. Here $\Delta \;=\; 1 - 4 \times (-2) \times (-1) \;=\; 1 - 8 \;<\; 0$.
Therefore let us complete the square.

$$-2x^2 + x - 1 = -2\left(x^2 - \frac{1}{2}x + \frac{1}{2}\right) = -2\left[\left(x - \frac{1}{4}\right)^2 + \frac{1}{2} - \frac{1}{16}\right] = -2\left[\left(x - \frac{1}{4}\right)^2 + \frac{7}{16}\right] < 0$$

Thus the function is negative for all values of *x*.

In example (i), $\Delta < 0$ and $a = 1 > 0 \Rightarrow$ the sign of the function is positive.
In example (ii), $\Delta < 0$ and $a = -3 < 0 \Rightarrow$ the sign of the function is negative.

Example (iii) $f(x) = 9x^2 + 6x + 1$. Here, $\Delta \;=\; 6^2 - 4 \times 9 \times 1 \;=\; 36 - 36 \;=\; 0$

$$9x^2 + 6x + 1 = (3x + 1)^2 \geq 0$$

$\therefore\; 9x^2 + 6x + 1 = 0$ when $x = -\dfrac{1}{3}$, and the function is positive for all other values of *x*.

Example (iv) $f(x) = -4x^2 + 4x - 1$. Here, $\Delta \;=\; 4^2 - 4 \times (-4) \times (-1) \;=\; 16 - 16 \;=\; 0$

$$-4x^2 + 4x - 1 = -(2x - 1)^2 \leq 0$$

$\therefore -4x^2 + 4x - 1 = 0$ when $x = \dfrac{1}{2}$, and the function is negative for all other values of *x*.

In example (iii), $\Delta = 0$ and $a = 9 > 0 \Rightarrow$ the function is non-negative.
In example (iv), $\Delta = 0$ and $a = -4 < 0 \Rightarrow$ the function is non-positive.

Example (v) $f(x) = 2x^2 + x - 1$. Here,

$$\Delta \;=\; 1^2 - 4 \times 2 \times (-1) \;=\; 1 + 8 \;=\; 9 \;>\; 0$$

This is positive and a perfect square. Therefore the function can be resolved into two rational factors.

$$2x^2 + x - 1 \;=\; (2x - 1)(x + 1) \;=\; 2\left(x - \frac{1}{2}\right)(x + 1)$$

$-1 < \dfrac{1}{2}$. Therefore, let us consider the intervals $x < -1$, $\;-1 < x < \dfrac{1}{2}$ and $x > \dfrac{1}{2}$.

	$x < -1$	$-1 < x < \frac{1}{2}$	$x > \frac{1}{2}$
$(x+1)$	-	+	+
$\left(x-\frac{1}{2}\right)$	-	-	+
$(x+1)\left(x-\frac{1}{2}\right)$	+	-	+
$2(x+1)\left(x-\frac{1}{2}\right)$	+	-	+

Therefore, the sign of the function is negative when $-1 < x < \frac{1}{2}$, and positive when $x < -1$ or $x > \frac{1}{2}$.

The zeros of the function are $x = -1$ and $x = \frac{1}{2}$.

Example (vi) $f(x) = -6x^2 + x + 2$

$\Delta = 1^2 - 4 \times (-6) \times 2 = 1 + 48 = 49 > 0$

This is a positive perfect square.

Therefore, the function can be resolved into two rational factors.

$-6x^2 + x + 2 = -\left(6x^2 - x - 2\right) = -(3x-2)(2x+1) = -6\left(x - \frac{2}{3}\right)\left(x + \frac{1}{2}\right)$

$-\frac{1}{2} < \frac{2}{3}$. Therefore, let us consider the intervals $x < -\frac{1}{2}$, $-\frac{1}{2} < x < \frac{2}{3}$ and $x > \frac{2}{3}$.

	$x < -\frac{1}{2}$	$-\frac{1}{2} < x < \frac{2}{3}$	$x > \frac{2}{3}$
$\left(x+\frac{1}{2}\right)$	-	+	+
$\left(x-\frac{2}{3}\right)$	-	-	+
$(x+\frac{1}{2})\left(x-\frac{2}{3}\right)$	+	-	+
$-6(x+\frac{1}{2})\left(x-\frac{2}{3}\right)$	-	+	-

Therefore, the sign of the function is positive when $-\frac{1}{2} < x < \frac{2}{3}$, and negative when $x < -\frac{1}{2}$ and when $x > \frac{2}{3}$. The zeros of the function are $x = -\frac{1}{2}$ and $x = \frac{2}{3}$.

4.1.2 The Maximum or Minimum Value of a Quadratic Function

$$ax^2 + bx + c = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right]$$

When $\Delta < 0$, $\quad ax^2 + bx + c = a\left[\left(x + \frac{b}{2a}\right)^2 + k\right], \quad k > 0$

For all values of x, $\left(x + \frac{b}{2a}\right)^2 \geq 0$.

If $a > 0$, $a\left(x + \frac{b}{2a}\right)^2 + ak \geq ak$, and when $a < 0$, $a\left(x + \frac{b}{2a}\right)^2 + ak \leq ak$.

That is, whenever $a > 0$, the value of $f(x) = ax^2 + bx + c$ is greater or equal to ak. $\therefore$ The minimum value that the function can take is ak.

Also, whenever $a < 0$, the value of $f(x) = ax^2 + bx + c$ is less than or equal to ak. $\therefore$ The maximum value that the function can take is ak.

$\therefore$ When $\Delta < 0$, if $a > 0$, the function has a minimum value and if $a < 0$, the function has a maximum value.

The maximum or minimum value is obtained when $x + \frac{b}{2a} = 0$; i.e., when $x = -\frac{b}{2a}$.

When $\Delta = 0$, $\quad ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2$

For all values of x, $\left(x + \frac{b}{2a}\right)^2 \geq 0$.

$\therefore \; a > 0 \Rightarrow a\left(x + \frac{b}{2a}\right)^2 \geq 0$ and $a < 0 \Rightarrow a\left(x + \frac{b}{2a}\right)^2 \leq 0$.

$\therefore$ When $\Delta = 0$, if $a > 0$, the function has a minimum value and if $a < 0$, the function has a maximum value.

The maximum or minimum value is obtained when $x + \frac{b}{2a} = 0$; and the value is zero.

When $\Delta > 0$,

$$ax^2 + bx + c = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right]$$

$\dfrac{\Delta}{4a^2} > 0$. $\therefore$ There exists $k \in \mathbb{R}$ such that $\dfrac{\Delta}{4a^2} = k^2$

For all values of x, $\left(x+\dfrac{b}{2a}\right)^2 \geq 0$

If $a > 0 \Rightarrow a\left(x+\dfrac{b}{2a}\right)^2 - ak^2 \geq -ak^2$ then the function has a minimum value.

If $a < 0 \Rightarrow a\left(x+\dfrac{b}{2a}\right)^2 - ak^2 \leq -ak^2$ then the function has a maximum value.

$\therefore$ When $\Delta > 0$, if $a > 0$, the function has a minimum value and if $a < 0$, the function has a maximum value.

The maximum or minimum value is obtained when $x+\dfrac{b}{2a} = 0$; i.e., when $x = -\dfrac{b}{2a}$.

In general,

- if $a > 0$, the function $f(x) = ax^2 + bx + c$ has a minimum value and if $a < 0$, the function has a maximum value. The maximum or minimum value is obtained when $x = -\dfrac{b}{2a}$.

Example (i) $f(x) = 3x^2 + 5x + 3$

Here $\Delta = 5^2 - 4\times3\times3 = 25-36 < 0$

$$3x^2+5x+3 = 3\left(x^2+\frac{5}{3}x+1\right) = 3\left[\left(x+\frac{5}{6}\right)^2 + 1 - \frac{25}{36}\right]$$

$$= 3\left[\left(x+\frac{5}{6}\right)^2 + \frac{11}{36}\right]$$

For all values of x, $\left(x+\dfrac{5}{6}\right)^2 \geq 0$

Adding $\dfrac{11}{36}$ to the two sides we obtain

$$\left(x+\frac{5}{6}\right)^2 + \frac{11}{36} \geq \frac{11}{36}$$

Multiplying both sides by 3 we obtain $3\left[\left(x+\dfrac{5}{6}\right)^2 + \dfrac{11}{36}\right] \geq 3\times\dfrac{11}{36} = \dfrac{11}{12}$

This implies that $3x^2+5x+3 \geq \dfrac{11}{12}$.

Therefore the minimum value that $f(x) = 3x^2+5x+3$ takes is $\dfrac{11}{12}$.

This minimum is achieved when $x+\dfrac{5}{6} = 0$; i.e., when $x = -\dfrac{5}{6}$.

Example (ii) $f(x) = -2x^2 + 3x - 5$.
Here $\Delta \;=\; 3^2 - 4\times(-2)\times(-5) \;=\; 9 - 40 \;=\; -31 \;<\; 0$

$$-2x^2 + 3x - 5 \;=\; -2\left(x^2 - \frac{3}{2}x + \frac{5}{2}\right) \;=\; -2\left[\left(x - \frac{3}{4}\right)^2 + \frac{5}{2} - \frac{9}{16}\right]$$
$$= \;-2\left[\left(x - \frac{3}{4}\right)^2 + \frac{40-9}{16}\right] \;=\; -2\left[\left(x - \frac{3}{4}\right)^2 + \frac{31}{16}\right]$$

For all values of *x*, $\left(x - \frac{3}{4}\right)^2 \;\geq\; 0$.

Adding $\frac{31}{16}$ to both sides we obtain, $\left(x - \frac{3}{4}\right)^2 \;+\; \frac{31}{16} \;\geq\; \frac{31}{16}$

Multiplying both sides by -2 we obtain

$$-2\left[\left(x - \frac{3}{4}\right)^2 \;+\; \frac{31}{16}\right] \;\leq\; -2\times\frac{31}{16} \;=\; -\frac{31}{8} \quad\Rightarrow\; -2x^2 + 3x - 5 \;\leq\; -\frac{31}{8}$$

Therefore, the maximum value that $f(x) = -2x^2 + 3x - 5$ can take is $-\frac{31}{8}$.

This maximum value is achieved when $x - \frac{3}{4} = 0$; i.e., when $x = \frac{3}{4}$.

Example (iii) $f(x) = 25x^2 - 10x + 1$.
Here $\Delta \;=\; 10^2 - 4\times 25\times 1 \;=\; 100 - 100 \;=\; 0$
$25x^2 - 10x + 1 \;=\; (5x-1)^2$

For all values of *x*, $(5x-1)^2 \geq 0$. Therefore, the minimum value that $f(x) = 25x^2 - 10x + 1$ can take is zero.

This minimum value is achieved when $5x - 1 = 0$; i.e., when $x = \frac{1}{5}$.

Example (iv) $f(x) = -9x^2 + 12x - 4$
Here $\Delta \;=\; 12^2 - 4\times(-9)\times(-4) \;=\; 144 - 144 \;=\; 0$
$-9x^2 + 12x - 4 \;=\; -\left(9x^2 - 12x + 4\right) \;=\; -(3x-2)^2$

For all values of *x*, $(3x-2)^2 \geq 0$. Therefore, $-(3x-2)^2 \leq 0$.

Hence the maximum value that $f(x) = -9x^2 + 12x - 4$ can take is 0. This is achieved when $3x - 2 = 0$; i.e., when $x = \frac{2}{3}$.

Example (v) $f(x)=2x^2+5x-3$. Here $\Delta=5^2-4\times2\times(-3)=25+24=49>0$

$$2x^2+5x-3=2\left(x^2+\frac{5}{2}x-\frac{3}{2}\right)=2\left[\left(x+\frac{5}{4}\right)^2-\frac{3}{2}-\frac{25}{16}\right]$$

$$=2\left[\left(x+\frac{5}{4}\right)^2-\frac{24+25}{16}\right]=2\left[\left(x+\frac{5}{4}\right)^2-\frac{49}{16}\right]$$

For all values of *x*, $\left(x+\frac{5}{4}\right)\geq 0$.

Adding $-\frac{49}{16}$ to both sides we obtain $\left(x+\frac{5}{4}\right)^2-\frac{49}{16}\geq-\frac{49}{16}$

Multiplying both sides by 2 we obtain $2\left[\left(x+\frac{5}{4}\right)^2-\frac{49}{16}\right]\geq-2\times\frac{49}{16}=-\frac{49}{8}$

The minimum value that $f(x)=2x^2+5x-3$ can take is $-\frac{49}{8}$.

This value is achieved by the function when $x=-\frac{5}{4}$.

Example (vi) $f(x)=-2x^2-3x+2$.
Here $\Delta=(-3)^2-4\times(-2)\times2=9+16=25>0$

$$-2x^2-3x+2=-2\left(x^2+\frac{3}{2}x-1\right)=-2\left[\left(x+\frac{3}{4}\right)^2-1-\frac{9}{16}\right]$$

$$=-2\left[\left(x+\frac{3}{4}\right)^2-\frac{25}{16}\right]=-2\left(x+\frac{3}{4}\right)^2+\frac{25}{8}$$

For all values of *x*, $\left(x+\frac{3}{4}\right)^2\geq 0$

Multiplying both sides by -2 we obtain $-2\left(x+\frac{3}{4}\right)^2\leq 0$

Adding $\frac{25}{8}$ to both sides we obtain $-2\left(x+\frac{3}{4}\right)^2+\frac{25}{8}\leq\frac{25}{8}$

Therefore, the maximum value that $f(x)=-2x^2-3x+2$ can take is $\frac{25}{8}$.

This is achieved when $x=-\frac{3}{4}$.

4.1.3 Symmetry

$$y = a\left[\left(x+\frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right]$$

The maximum or minimum value of y is achieved when $x = -\frac{b}{2a}$.

Let us consider again the above examples.

(i) $y = 3x^2 + 5x + 3$. $y_{\min} = \frac{11}{12}$.

Then, $x = -\frac{5}{6} \Rightarrow \left(-\frac{5}{6}, \frac{11}{12}\right)$ is the minimum point.

The graph of the function is symmetric about the line $x = -\frac{5}{6}$.

$\therefore$ the axis of symmetry is $x = -\frac{5}{6}$.

In the same manner, the axes of symmetry for the other examples are obtained as follows.

(ii) $x = \frac{3}{4}$ (iii) $x = \frac{1}{5}$ (iv) $x = \frac{2}{3}$ (v) $x = -\frac{5}{4}$ (vi) $x = -\frac{3}{4}$.

- ♦ In general, the axis of symmetry of the function $f(x) = ax^2 + bx + c$ is $x = -\frac{b}{2a}$.

4.1.4 Existence of Real Zeros

If there exists a real value of x such that $ax^2 + bx + c = 0$, we say that the function $y = ax^2 + bx + c$ has a real zero.

It has been shown previously that when $\Delta < 0$, $y = ax^2 + bx + c$ can be expressed as

$$y = a\left[\left(x+\frac{b}{2a}\right)^2 + k\right], \quad k > 0.$$

Since $\left(x+\frac{b}{2a}\right)^2 \geq 0$ and $k > 0$, y cannot be zero for any value of x.

Thus when $\Delta < 0$, the function $y = ax^2 + bx + c$ has no real zeros.

When $\Delta = 0$, the function $y = ax^2 + bx + c$ has a real zero $x = -\frac{b}{2a}$.

When $\Delta > 0$, $y = ax^2 + bx + c$ can be factored as $y = a(x-\alpha)(x-\beta)$, where the factors are real. $y = 0$ when $x = \alpha$ or $x = \beta$. $\therefore$ when $\Delta > 0$, y has two distinct real zeros.

$\therefore$ When $\Delta \geq 0$, $y = ax^2 + bx + c$ has real zeros and when $\Delta < 0$, y has no real zeros.

4.1.5 The Graphs of Quadratic Functions

When sketching the graph of a quadratic function, find

- the maximum or minimum point
- axis of symmetry
- the points at which the graph intersects the axes.

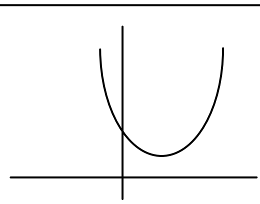
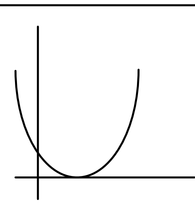
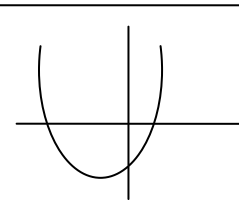
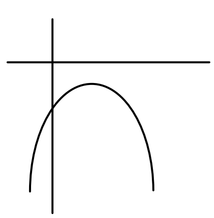
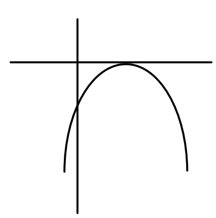
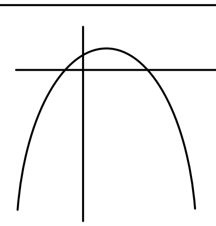
Now, the graphs of the functions considered in examples (i) to (vi) can be sketched.
Let us complete the following table with the information obtained.

Function	**Sign of Δ**	**Sign of *a***	**Sign of *y***	**Nature of the graph Maximum/Minimum**	**Intersects the *x*-axis or not**
(i) $y = 3x^2 + 5x + 3$	-	+	+	Minimum	Does not
(ii) $y = -2x^2 + 3x - 5$	-	-	-	Maximum	Does not
(iii) $y = 25x^2 - 10x + 1$	$\Delta = 0$	+	+ or $y = 0$	Minimum	Touches
(iv) $y = -9x^2 + 12x - 4$	$\Delta = 0$	-	- or $y = 0$	Maximum	Touches
(v) $y = 2x^2 + 5x - 3$	+	+	+ or –	Minimum	Does
(vi) $y = -2x^2 - 3x + 2$	+	-	+ or -	Maximum	Does

From the above table we can come to the following conclusions.

- $a > 0 \Rightarrow$ the graph has a minimum
- $a < 0 \Rightarrow$ the graph has a maximum
- $\Delta < 0 \Rightarrow$ the graph does not intersect the x - axis
- $\Delta = 0 \Rightarrow$ the graph touches the x - axis
- $\Delta > 0 \Rightarrow$ the graph intersects the x - axis at two real distinct points

Therefore, the graph of the function $y = ax^2 + bx + c$ can be summarized as follows for the various cases.

	$\Delta < 0$	$\Delta = 0$	$\Delta > 0$
$a > 0$			
$a < 0$			

Exercises

1. Discuss the sign of the following functions

   (i) $f(x)=x^2-4x+7$ (ii) $f(x)=-x^2+10x-26$ (iii) $f(x)=x^2-6x+9$
   (iv) $f(x)=14x-x^2-49$ (v) $f(x)=x^2+3x-18$ (vi) $f(x)=9x^2+6x+2$
   (vii) $f(x)=12x-4x^2-1$ (viii) $f(x)=49x^2+14x+1$
   (ix) $f(x)=12x-36x^2-1$ (x) $f(x)=1-2x-3x^2$
   (xi) $f(x)=a^2x^2+2abx+b^2$ (xii) $f(x)=-\frac{1}{4}+2x-4x^2$

2. Determine the range of values of k for which the function $f(x)=2x^2+kx+k>0$.

3. Show that for the expression $-9+4kx-x^2$ to be negative for all real values of x, the values that k can assume should lie between $-\frac{3}{2}$ and $\frac{3}{2}$.

4. Determine the range of values of k for which $x^2+(k-1)x+k+2>0$ for all real values of x.

5. Determine the range of values of p for which $f(x)=x^2+3px+p$ is positive.

6. $f(x)=5x^2-10x-k$. Determine the range of values of k for which $f(x)>2$ for all real values of x.

7. For which range of values of x is the expression $(x^2-4x-21)(12-25x+12x^2)$ negative?

8. Determine the maximum or minimum value of the following functions which are defined for all real values of x. Determine also the value of x which gives the maximum or minimum value.

   (i) $f(x)=x^2-4x+5$ (ii) $f(x)=8-12x-3x^2$
   (iii) $f(x)=3x-x^2$ (iv) $f(x)=1-4x-3x^2$

9. $f(x)=3x^2-px-p$. For which value of p is the minimum value of $f(x)$ equal to zero?

10. Show that the function $f(x)=x^2+2a^2+2b^2-2bx+2ax-4ab$ can never be negative. Deduce that the minimum value of the function is achieved when $x = b - a$, and that the minimum value it takes is $(a-b)^2$.

11. The two functions f and g are given by $f(x) = x^2 + 20x + 2p^2$ and $g(x) = -x^2 - 2x + 27$. Determine the minimum value of $f(x)$ and the maximum value of $g(x)$. Determine the value of p for which the minimum value of $f(x)$ > the maximum value of $g(x)$.

12. Prove that the minimum value of the quadratic function $y = ax^2 + bx + c$ for $a > 0$ is $-\frac{\Delta}{4a}$.

A quadratic function of x takes the value zero when $x = 1$ and the value 5 when $x = 0$. If the minimum value of the function is -4, find the function.

13. Prove that the maximum value of the quadratic function $y = ax^2 + bx + c$ for $a < 0$ is $-\frac{\Delta}{4a}$.

A quadratic function of x takes the value zero when $x = 2$ and has a maximum value of $\frac{1}{4}$. The value of the function when $x = 0$ is -2. Determine the relevant function.

14. Determine the maximum or minimum value of each of the following functions and sketch their graphs.

(i) $f(x) = x^2 - 5x + 6$ (ii) $f(x) = -x^2 - 4x + 21$

(iii) $f(x) = 2x^2 - x$ (iv) $f(x) = -4x^2 - 9$ (v) $f(x) = 9 - 3x - 2x^2$

15. Sketch the graph of $f(x) = x^2 - 2x + 5$. Hence sketch the graphs of

(i) $y_1 = f(x+1)$ (ii) $y_2 = f(x) + 1$

For each function, write down the minimum value and the equation of the axis of symmetry.

4.2 Quadratic Equations

4.2.1 The General Form of a Quadratic Equation

An equation of the form $ax^2 + bx + c = 0$ where a, b, c are real constants such that $a \neq 0$, is defined to be a quadratic equation.

4.2.2 Finding the Roots of a Quadratic Equation

(i) By resolving into factors
(ii) By completing the square
(iii) By using the formula

♦ The method of completing the square or the formula can be used to find the roots when the quadratic expression cannot be resolved into factors easily.

4.2.3 The Nature of Quadratic Equations

The roots of $ax^2 + bx + c = 0$ are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
Here $\Delta = b^2 - 4ac$.

♦ If $\Delta > 0$ and a perfect square, the roots are real and distinct. If the coefficients of the quadratic equation are integers, then the roots are rational.
If $\Delta > 0$ and not a perfect square, the roots are distinct and irrational.
If $\Delta = 0$, the roots are real and equal. If the coefficients of the quadratic equation are integers, then the roots are rational.
If $\Delta < 0$, the roots are complex.

Example (i)
$2x^2 - 7x + 3 = 0$. Here $\Delta = 7^2 - 4 \times 2 \times 3 = 49 - 24 = 25$.
$\therefore \Delta > 0$ and a perfect square. Also, the coefficients of the quadratic equation are integers. Therefore, the roots should be distinct and rational.

$$2x^2 - 7x + 3 = 0 \Rightarrow x = \frac{7 \pm \sqrt{25}}{2 \times 2} = \frac{7 \pm 5}{4} = 3 \text{ or } \frac{1}{2}.$$

Therefore, the roots are rational and unequal.

(ii) $x^2 + x - 1 = 0$. Here $\Delta = 1^2 - 4 \times 1 \times (-1) = 1 + 4 = 5$
$\therefore \Delta > 0$, but is not a perfect square.
Therefore, the roots should be irrational and distinct.

$$x^2 + x - 1 = 0 \Rightarrow x = \frac{-1 \pm \sqrt{5}}{2 \times 1} = \frac{-1 \pm \sqrt{5}}{2}$$

$\therefore x = \dfrac{-1 + \sqrt{5}}{2}$ or $\dfrac{-1 - \sqrt{5}}{2}$. The roots are distinct and irrational.

(iii) $4x^2 + 12x + 9 = 0$. Here $\Delta = 12^2 - 4\times 4\times 9 = 144 - 144 = 0$

$$4x^2 + 12x + 9 = 0 \Rightarrow (2x+3)^2 = 0 \Rightarrow x = -\frac{3}{2}$$

Therefore the roots are real, rational and equal.

(iv) $x^2 + x + 1 = 0$. Here, $\Delta = 1^2 - 4\times 1\times 1 = 1 - 4 = -3$

When we complete the square we obtain

$$\left(x+\frac{1}{2}\right)^2 - \frac{1}{4} + 1 = 0 \Rightarrow \left(x+\frac{1}{2}\right)^2 + \frac{3}{4} = 0 \Rightarrow \left(x+\frac{1}{2}\right)^2 = -\frac{3}{4}$$

Since $\left(x+\frac{1}{2}\right)^2$ is always non-negative, this is not possible. Therefore, there are no real values of x satisfying the equation. Therefore, the equation has no real roots.

4.2.4 The Relationships Between the Roots of a Quadratic Equation

Since $ax^2 + bx + c = 0$ with $a \neq 0$, $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$...............$(1)$

If the roots of this equation are α and β, the equation can be written as $(x-\alpha)(x-\beta) = 0$

That is, as $x^2 - (\alpha+\beta)x + \alpha\beta = 0$...............$(2)$

(1) and (2) both represent the same equation.

By equating the coefficients we obtain

$\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$.

The sum of the roots = $-\dfrac{\text{the coefficient of } x}{\text{the coefficient of } x^2}$

The product of the roots = $\dfrac{\text{the constant coefficient}}{\text{the coefficient of } x^2}$

Example 1: Determine the sum and the product of the roots of the following equations

(i) $2x^2 + 3x + 5 = 0$ (ii) $3x^2 - x + 1 = 0$

(i) Sum of the roots = $-\frac{3}{2}$, product of the roots = $\frac{5}{2}$

(ii) Sum of the roots = $\frac{1}{3}$, product of the roots = $\frac{1}{3}$

Example 2: If α and β are the roots of the equation $ax^2+bx+c=0$ with $a\neq 0$, determine the following.

(i) $\alpha^2+\beta^2$ (ii) $\alpha^3+\beta^3$

α and β are the roots of the equation $ax^2+bx+c=0$

$\therefore\ \alpha+\beta=-\dfrac{b}{a}$ and $\alpha\beta=\dfrac{c}{a}$

(i) $\alpha^2+\beta^2 = (\alpha+\beta)^2-2\alpha\beta = \left(-\dfrac{b}{a}\right)^2-2\dfrac{c}{a} = \dfrac{b^2}{a^2}-\dfrac{2c}{a} = \dfrac{b^2-2ac}{a^2}$

(ii) $\alpha^3+\beta^3 = (\alpha+\beta)(\alpha^2+\beta^2-\alpha\beta) = (\alpha+\beta)\left[(\alpha+\beta)^2-2\alpha\beta-\alpha\beta\right]$

$$= (\alpha+\beta)\left[(\alpha+\beta)^2-3\alpha\beta\right] = \left(-\frac{b}{a}\right)\left[\left(-\frac{b}{a}\right)^2-3\frac{c}{a}\right]$$

$$= \left(-\frac{b}{a}\right)\left[\frac{b^2-3ac}{a^2}\right] = -\frac{b\left(b^2-3ac\right)}{a^3}$$

Suppose α and β are the roots of the equation $ax^2+bx+c=0$

Then $\alpha+\beta=-\dfrac{b}{a}$ and $\alpha\beta=\dfrac{c}{a}$

When both the roots are positive, $\alpha>0,\ \beta>0 \Rightarrow \alpha+\beta>0$ and $\alpha\beta>0$.

This implies that $-\dfrac{b}{a}>0$ and $\dfrac{c}{a}>0$.

$-\dfrac{b}{a}>0 \Rightarrow$ *a* and *b* have opposite signs.

$\dfrac{c}{a}>0 \Rightarrow$ *a* and *c* have the same sign.

Therefore, when both the roots are positive, *a* and *c* have the same sign while *b* has the opposite sign.

When both the roots are negative, $\alpha<0,\ \beta<0 \Rightarrow \alpha+\beta<0$ and $\alpha\beta>0$.

This implies that $-\dfrac{b}{a}<0$ and $\dfrac{c}{a}>0$.

$-\dfrac{b}{a}<0 \Rightarrow$ *a* and *b* have the same sign.

$\frac{c}{a} > 0 \Rightarrow a$ and c have the same sign.

Therefore, when both the roots are negative, a, b and c have the same sign.

When the roots have opposite signs, then either $\alpha < 0,\ \beta > 0$ or $\alpha > 0,\ \beta < 0$.

Then, $\alpha\beta < 0 \Rightarrow \frac{c}{a} < 0 \Rightarrow a$ and c have opposite signs.

When the roots are numerically equal but of opposite signs:
When the roots are of opposite signs, a and c are of opposite signs.
When the roots are numerically equal and of opposite signs, $\beta = -\alpha$

$\beta = -\alpha \Rightarrow \alpha + \beta = 0 \Rightarrow -\frac{b}{a} = 0 \Rightarrow b = 0$

$\therefore$ when the roots are numerically equal but of opposite signs, a and c are of opposite signs and $b = 0$.

When one of the roots is zero then either $\alpha = 0$ or $\beta = 0$

$\alpha = 0$ or $\beta = 0 \Rightarrow \alpha\beta = 0 \Rightarrow \frac{c}{a} = 0 \Rightarrow c = 0$

4.2.5 Constructing Quadratic Equations when the Roots are given

When the roots of an equation are given as α and β, the equation can be written as
$(x-\alpha)(x-\beta) = 0$

i.e., as $x^2 - (\alpha+\beta)x + \alpha\beta = 0$
$\therefore$ the quadratic equation is: $x^2 - (\text{the sum of the roots})x + (\text{the product of the roots}) = 0$

Example: The quadratic equation with roots equal to 3 and 2:
$x^2 - (3+2)x + 3\times 2 = 0 \Rightarrow x^2 - 5x + 6 = 0$

4.2.6 Constructing Quadratic Equations with Roots which are Symmetric Expressions of Roots of a given Quadratic Equation

If α and β are the roots of the equation $ax^2 + bx + c = 0$ with $a \neq 0$, determine the sum and the product of the roots of the required equation in terms of a, b and c.
Then the quadratic equation is given by
$x^2 - (\text{the sum of the roots})x + (\text{the product of the roots}) = 0$

4.2.7 Deducing Quadratic Equations with Roots which are Symmetric Expressions of Roots of a given Quadratic Equation

If α and β are the roots of the equation $ax^2 + bx + c = 0$ with $a \neq 0$, let us deduce the equation with has as its roots $\frac{1}{\alpha}$ and $\frac{1}{\beta}$.

The roots of the equation $ax^2 + bx + c = 0$ are the values of x which satisfy the equation. Since α and β are the roots of the given equation $x = \alpha$ and $x = \beta$.

The equation that has as its roots $\frac{1}{\alpha}$ and $\frac{1}{\beta}$, is the equation which $\frac{1}{x}$ satisfies.

Let us substitute $y = \frac{1}{x}$.

When $y = \frac{1}{x}$,

$$a\left(\frac{1}{y}\right)^2 + b\left(\frac{1}{y}\right) + c \ = \ 0 \ \Rightarrow \ cy^2 + by + a \ = \ 0$$

Since $y = \frac{1}{x}$ and $x = \alpha$ and $x = \beta$, $y = \frac{1}{\alpha}$ or $y = \frac{1}{\beta}$.

The equation which has $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ as its roots is $cx^2 + bx + c = 0$

Exercises

1. Without solving the equation, determine whether the roots of the following equations are real, distinct, equal or unequal.

   (i) $x^2 - 7x + 9 = 0$ (ii) $2x^2 - 5x + 3 = 0$ (iii) $4x^2 - 12x + 11 = 0$

   (iv) $2x^2 - 9x + 6 = 0$ (v) $x^2 - 2ax + a^2 = 0$ (vi) $(2 + x)^2 = (3x - 1)^2$

2. If a, b and c are rational numbers, show that the roots of the equation $x^2 - 2ax + a^2 - b^2 + 2bc - c^2 = 0$ are rational.

3. Prove that for all values of k , the roots of each of the following equations are real.

   (i) $x^2 + 2(k+1)x - (2k+3) = 0$ (ii) $(k-3)x^2 + 2kx + 1 = 0$

   (iii) $kx^2 - (3k-2)x + k - 1 = 0$ (iv) $(k^2+1)x^2 - (2k-1)x - 3 = 0$

4. If p and q are real numbers such that $p \neq q$, prove that the roots of the following equation are not real: $x^2 - 2px + (2p^2 - 2pq + q^2) = 0$

5. $f(x) = x^2 + (b+2)x + 2b$. If $2a = 2 - b$, show that the roots of $f(x-b) + c^2 = 0$ are equal.

6. Determine the range of values of k for which each of the following equations has real roots.

   (i) $(x-3)(x-5) + k^2 = 0$ (ii) $x^2 - kx + k = 0$

   (vii) $(5k-1)x^2 + kx = 1 - 5k$ (viii) $x^2 - 2(1+3k)x + 7(3+2k) = 0$

7. If the roots of each of the following equations are equal, determine k.

   (i) $kx^2 - 16x + 4k = 0$ (ii) $4x^2 + 2kx + 3 - k = 0$

8. If the roots of the equation $ax^2 + bx + c = 0$ are imaginary, show that the roots of the following equation are also imaginary: $ax^2 - 2(a+b)x + (a + 2b + 4c) = 0$

9. Determine the sum and the product of the roots of each of the following quadratic equations

   (i) $x^2 - 5x + 3 = 0$ (ii) $x^2 + 9x + 3 = 0$ (iii) $5x^2 - 10x - 1 = 0$

   (iv) $2x^2 + 3x - 3 = 0$ (v) $ax^2 - (a+1)x - a = 0$ (vi) $\frac{x-1}{2} = \frac{x-3}{x+2}$

10. If α and β are the roots of the equation $2x^2 - x + 3 = 0$, determine the following:

    (i) $\alpha^4 + \beta^4$ (ii) $(\alpha+1)(\beta+1)$ (iii) $\left(\alpha + \frac{1}{\alpha}\right)\left(\beta + \frac{1}{\beta}\right)$

11. If α and β are the roots of the equation $ax^2 + bx + c = 0$, prove that $a\left(\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}\right) + b\left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right) = b$

12. If one root of the equation $kx^2 + x + 1 = 0$ is k times the root of the other equation, determine the values that k can take.

13. If the roots of the equation $(a+x)(b+x) - c(a+x) - d(b+x) = 0$ are α and β, prove that $(\alpha - \beta)^2 = (a - b + c - d)^2 + 4cd$

Deduce that if a, b, c and d are real and c and d are either both positive or both negative, then α and β are both real.

14. If the roots of $3x^2 - 5x + 7 = 0$ are α and β, prove that $3(\alpha^5 + \beta^5) - 5(\alpha^4 + \beta^4) + 7(\alpha^3 + \beta^3) = 0$

15. If the roots of $x^2 - px - q = 0$ are α and β, prove that $\alpha^n + \beta^n = p(\alpha^{n-1} + \beta^{n-1}) = q(\alpha^{n-2} + \beta^{n-2})$

Hence determine the value of $\alpha^5 + \beta^5$, if the roots of $x^2 - 2x - 1 = 0$ are α and β.

16. α and β are the roots of the equation $9x^2 + 6x + 1 = 4\lambda x$, where λ is a real constant. For the of the following cases, determine the range of λ.

(i) When both α and β are positive.
(ii) When both α and β are negative.

17. Determine the condition that has to be satisfied for the roots of the equation $\frac{a}{x-a} + \frac{b}{x-a} = 1$ to be numerically equal but opposite in sign.

18. If the roots of the equation $\frac{x^2 - bx}{ax - c} = \frac{m-1}{m+1}$ are numerically equal but opposite in sign, show that $m = \frac{a-b}{a+b}$.

19. Show that if the equations $x^2 + bx + c = 0$ and $x^2 + cx + b = 0$ have a common root, then either $b = c$ or $b + c + 1 = 0$.

20. If each of the following pairs of equations have a common root, determine the value of k.
(i) $x^2 + 2x + 3k = 0$ and $2x^2 + 3x + 5k = 0$
(ii) $3x^2 + 4kx + 2 = 0$ and $2x^2 + 3x - 2 = 0$

21. Construct the quadratic equation having the following roots.
(i) 2 and 5 (ii) 4 and -1 (ii) 3 and $\frac{1}{3}$

22. If α and β are the roots of $3x^2+2x+1=0$, construct the quadratic equation with roots $\dfrac{1-\alpha}{1+\alpha}$ and $\dfrac{1-\beta}{1+\beta}$.

23. If α and β are the roots of $2x^2-3x+1=0$, construct the quadratic equations with the following roots.

(i) α^2 and β^2 (ii) α^3 and β^3

(iii) $\dfrac{1}{\alpha^2}$ and $\dfrac{1}{\beta^2}$ (iv) $\dfrac{\alpha}{2\beta+3}$ and $\dfrac{\beta}{2\alpha+3}$

24. If α and β are the roots of the equation $ax^2+bx+c=0$, construct the quadratic equation with roots α^2 and β^2. Hence deduce the quadratic equation with roots $\dfrac{1}{\alpha^2}$ and $\dfrac{1}{\beta^2}$.

25. If α and β are the roots of the equation $ax^2+bx+c=0$, construct the quadratic equation with roots α^3 and β^3. Hence deduce the quadratic equation with roots $\dfrac{\alpha^3}{\beta^3}+1$ and $\dfrac{\beta^3}{\alpha^3}+1$.

5. Vectors

Competency 1	:	Manipulates Vector Algebra
Competency Level	:	1.1, 1.2, 1.3, 1.4
Subject Part	:	Vectors (Teachers Guide, Grade 12 - Pages 20-28)

By studying this section you will develop the skills of

- To explain the difference between scalar quantities and scalars.
- To define vector quantities.
- To represent a vector geometrically.
- To present a vector algebraically.
- To classify vectors.
- To define the modulus of a vector.
- To express the conditions for the equality of two vectors.
- To define the reverse vector.
- To express the triangle law of vector addition.
- To deduce the parallelogram law of vector addition.
- To subtract two vectors.
- To define the null vector.
- To multiply a vector by a scalar.
- To introduce the angle between two vectors.
- To introduce parallel vectors.
- To define the unit vector.
- To state the conditions for two vectors to be parallel.
- To add three or more vectors.
- To state the properties of vector addition.
- To resolve a vector in any two given directions.
- To resolve vectors in two directions perpendicular to each other.
- To introduce the position vector.
- To express a given vector in the form $x\underline{i} + y\underline{j}$
- To define the scalar product of two vectors
- To state that the scalar product of two vectors is a scalar.
- To state the properties of scalar product.
- To represent a scalar product geometrically.
- To solve simple geometrical problems using the scalar product.
- To define the vector product of two vectors.
- To state that the vector product of two vectors is a vector.
- To state the properties of vector product.

Introduction

It has occured to the human mind from antiquity that not only the magnitude but also the direction should be taken into account in the communication of certain mathematical quantities encountered in various activities of the numan society. Later such quantities were called vector quantities and were formally defined.

The triangle law of vector addition and hence the parallelogram law of vector addition were introduced in terms of the practical situations of the vector quantity, displacement.

Later the scalar product and vector product of two vectors were defined in agreement with certain phenomena in Physical Science and as a tool of solving various mathematical problems conveniently.

5.1 Scalars and scalar quantities

Real numbers such as $0,\ -2.1,\ 8\frac{1}{3},\ \sqrt{5},\ \ldots$ without a unit, but with a magnitude only are known as scalars. Quantities given with some unit and which have a magnitude only are knwon as scalar quantities and numerical quantities without a unit are known as scalars.

Examples :

Scalars : Numerical values 10, 25, 12, 50 are scalars.

Scalar quantities : (i) length 11.5 m (ii) Mass 25 kg

(iii) Temperature 30^0 C (iv) 25 m^3

5.2 Vector quantities

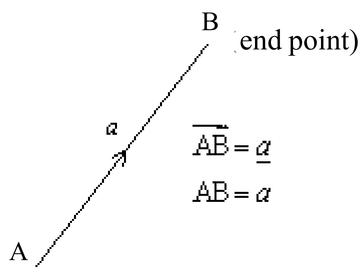
A quantity with a magnitude and a direction and which agrees with the triangle law (or parallelogram law) of addition is known as a vector quantity. The triangle law and the parallelogram law on the addition of vectors will be introduced later.

Examples :

i. Displacement is 10km due north ii. Velocity 15 m s^{-1} 30 west of South

iii. Weight is 28kg vertically downwards iv. A force 10 N inclined at 60° with the upward vertical

5.3 Geometrical Representation



A line segment with a magnitude and a direction to a scale parallel to a vector is known as a geometrical vector. The magnitude of the vector is denoted by its length and the direction of the vector is denoted by its direction and an arrowhead on it.

Examples :

Denote the following vectors by line segments.

i. A displacement of 10 m due east.
ii. A displacement of 100 m due north.
iii. A velocity of 50 m per second due east.
iv. A horizontal force of 100 N.

5.4 Algebraic Representation

Vectors and vector quantities can be represented algebraically by sumbols such as $\overrightarrow{AB}$, $\overrightarrow{CD}$, $\overrightarrow{PQ}$, $\overrightarrow{DE}$, ... or by $\underline{a}, \underline{b}, \underline{r}, \ldots$ or by $\vec{a}, \vec{b}, \vec{r}, \ldots$ or by $\vec{A}$, $\vec{B}$, $\vec{F}$, $\vec{R}$, ... In print they are denoted by bold black letters. Then an arrowhead above or below the letter is not necessary.

5.5 Classification of Vectors

1. Free Vectors

Vectors with only a magnitude and direction and without a definite point or line of action are known as free vectors. (Examples : Vectors used in addition law).

2. Sliding Vectors

Vectors with a magnitude, direction and a definite line of action and without a definite point of action are known as sliding vectors. (For example : forces applied on rigid body)

3. Tied Vectors

Vectors with a magnitude, direction, a definite line of action and a definite point of action are known as tied vectors. (For example : forces applied on a particle).

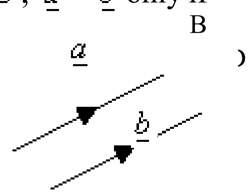
5.6 Modulus of Vectors

The magnitude of a vector is known as its modulus. The modulus of the vector is denoted by $|\underline{a}|$ and the modulus of the vector $\overrightarrow{AB}$ is denoted by $|\overrightarrow{AB}|$ or AB. $|\underline{a}|$ or $|\overrightarrow{AB}|$ are read as the modulus of vector " $\underline{a}$ or "the modulus of the vector $\overrightarrow{AB}$. The modulus of a vector is never negative. That is the modulus of vector is always either zero or positive.

5.7 Equality of two Vectors

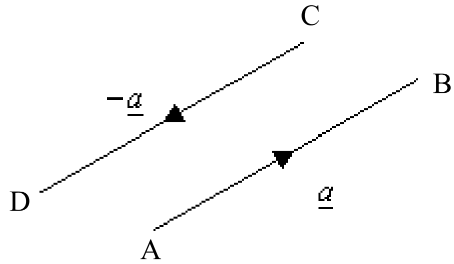
Vectors with equal magnitude and in the same direction are known as equal vectors. When the vector $\underline{a}$ is represented by $\overrightarrow{AB}$ and the vector $\underline{b}$ is represented by $\overrightarrow{CD}$, $\underline{a} = \underline{b}$ only if

i. AB = CD (magnitudes are equal)
ii. AB // CD (directions are equal)
iii. Sense from A to B = Sense from C to D



"Two vectors are in the same direction" implies that the two vectors are parallel and are on the same line."

5.8 Negative Vector (Reverse Vector)



When the vector $\underline{a}$ is represented by $\overrightarrow{AB}$,

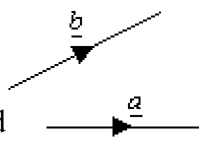
$\overrightarrow{CD} = -\underline{a}$ only if

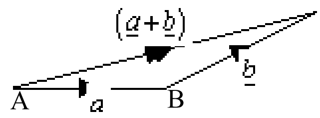
(i) CD = AB (ii) CD // AB

(iii) Sense fron C to D = Sense from B to A.

5.9 Vector Addition

5.9.1 Triangle Law





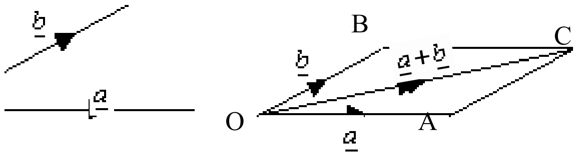
$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

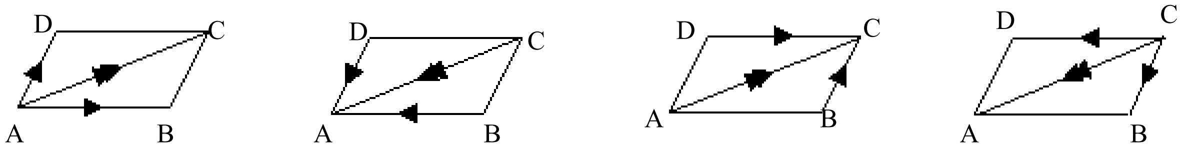
When any two vectors $\underline{a}$ and $\underline{b}$ of the same type are represented by $\overrightarrow{AB}$ and $\overrightarrow{BC}$ respectively, the sum of $\underline{a}$ and $\underline{b}$ $(\underline{a}+\underline{b})$ is represented by AC of triangle ABC .

For example :- $\overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{BD}$, $\overrightarrow{CD} + \overrightarrow{DE} = \overrightarrow{CE}$,

5.9.2 Parallelogram Law

When any two vectors $\underline{a}$, $\underline{b}$ of the same type are represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively, the sum of $\underline{a}$ and $\underline{b}$ is represented by $\overrightarrow{OC}$ of the parallelogram OACB. When ABCD is a parallelogram

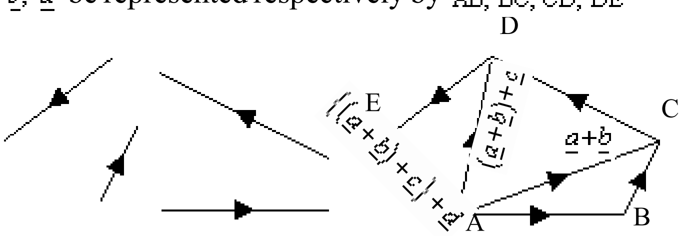




$\overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AC}$ $\overrightarrow{BA} + \overrightarrow{DA} = \overrightarrow{CA}$ $\overrightarrow{DC} + \overrightarrow{BC} = \overrightarrow{AC}$ $\overrightarrow{CB} + \overrightarrow{CD} = \overrightarrow{CA}$

5.9.3 Sum of several Vectors

Let the vectors $\underline{a}$, $\underline{b}$, $\underline{c}$, $\underline{d}$ be represented respectively by $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$, $\overrightarrow{DE}$



$$\{(\underline{a}+\underline{b})+\underline{c}\}+\underline{d} = \{(\overrightarrow{AB}+\overrightarrow{BC})+\overrightarrow{CD}\}+\overrightarrow{DE}$$

$$= \{\overrightarrow{AC}+\overrightarrow{CD}\}+\overrightarrow{DE} = \overrightarrow{AD}+\overrightarrow{DE}$$

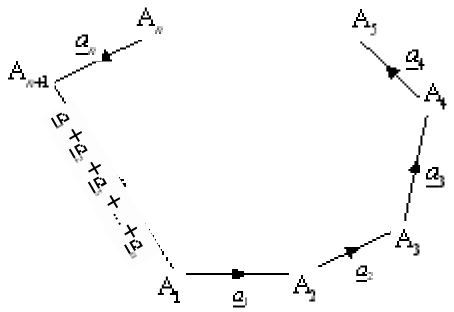
$$= \overrightarrow{AE}$$

Any number of vectors can be added by this method. The result obtained by successive application of triangle law for addition of vectors is known as the polygon law.

5.9.4 Polygon Law of Vector addition

If the vectors $\underline{a}_1, \underline{a}_2, \underline{a}_3, \ldots\ldots, \underline{a}_n$ are represented by the sides $A_1A_2A_3 \ldots A_nA_{n+1}$ of the polygon $\overrightarrow{A_1A_2}, \overrightarrow{A_2A_3}, \overrightarrow{A_3A_4}, \ldots, \overrightarrow{A_nA_{n+1}}$

$\underline{a}_1 + \underline{a}_2 + \underline{a}_3 + \ldots.. + \underline{a}_n$ is represented by the vector $\overrightarrow{A_1A_{n+1}}$ Here it is not necessary for the polygon obtained to be convex.



5.10 Subtraction of Vectors

When $\underline{a}$ and $\underline{b}$ are any two vectors, subtracting this vector $\underline{b}$ from the vector $\underline{a}$ is the same as adding the vector $-\underline{b}$ to the vector $\underline{a}$

Method 1

When the vector $\underline{a}$ is represented by $\overrightarrow{OA}$ and the vector $\underline{b}$ by $\overrightarrow{OB}$, $(\underline{a}-\underline{b})$ is represented by $\overrightarrow{BA}$

Proof:

$\underline{a}-\underline{b} = \underline{a}+(-\underline{b}) = \overrightarrow{OA}+(-\overrightarrow{OB}) = \overrightarrow{OA}+\overrightarrow{BO} = \overrightarrow{BO}+\overrightarrow{OA}$ (Commulative Law) $= \overrightarrow{BA}$

$(\underline{a}-\underline{b})$ is represented by $\overrightarrow{AB}$

5.11 Null Vectors

A vector with zero magnitude and an arbitrary direction is known as a null vector. It is denoted by $\underline{0}$. $|\underline{0}| = 0$ The null vector can be represented geometrically by a point.

$|\overrightarrow{AA}| = AA = 0$

The null vector if represented by $\therefore \overrightarrow{AA}$

5.12 Multiplication of a Vector by a Scalar

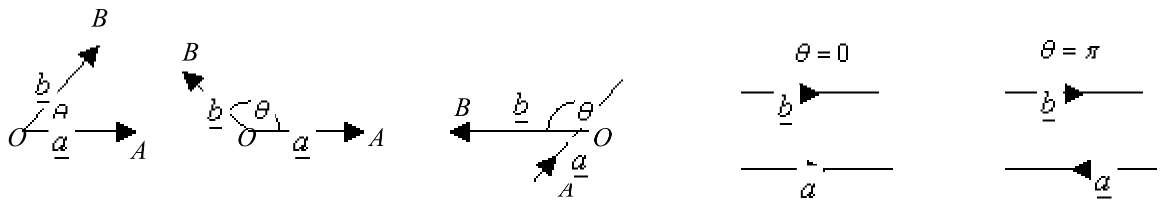
When λ is any scalar and $\underline{a}$ is any vector $\lambda\underline{a}$ is

i. The vector $\lambda|\underline{a}|$ in the direction of $\underline{a}$ when $\lambda > 0$

ii. The vector $-\lambda|\underline{a}|$ in the direction opposite to that of $\underline{a}$ when $\lambda < 0$

iii. The null vector when $\lambda = 0$ That is $0\underline{a} = 0$

5.13 Angle between two Vectors



When a vector $\underline{a}$ is represented by $\overrightarrow{OA}$ and a vector $\underline{b}$ by $\overrightarrow{OB}$ and if the angle between the direction $\overrightarrow{OA}$ and $\overrightarrow{OB}$ is defined as the angle between the two vectors. If the angle between the two vectors $\underline{a}$ and $\underline{b}$ is θ, $0 \le \theta \le \pi$.

5.14 Parallel Vectors

When a vector $\underline{a}$ is represented by $\overrightarrow{AB}$ and a vector $\underline{b}$ by $\overrightarrow{CD}$, $\underline{a} // \underline{b}$ only if $AB // CD$.

i. When $\underline{a} // \underline{b}$, $\underline{a}$ and $\underline{b}$ are in the same direction only if $\underline{a} = k\underline{b}$ and $k < 0$

ii. When $\underline{a} // \underline{b}$, $\underline{a}$ and $\underline{b}$ are in opposite direction only if $\underline{a} = k\underline{b}$ and $k < 0$ $\underline{a} // \underline{b}$

5.15 Unit Vector

Vectors with magnitude of one unit and which agree with the triangle law or parallelogram law of vector addition are known as unit vectors. If $\underline{l}$ is a unit vector then $|\underline{l}| = 1$

Conversely if $|\underline{l}| = 1$ then is a unit vector.

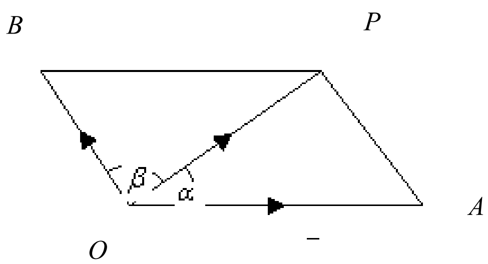
5.16 Resolution of Vectors

Expressing a given vector as a sum of two or more vectors is known as resolving the given vector the vectors appearing in the sum are called the resolved parts or components of the given vector.

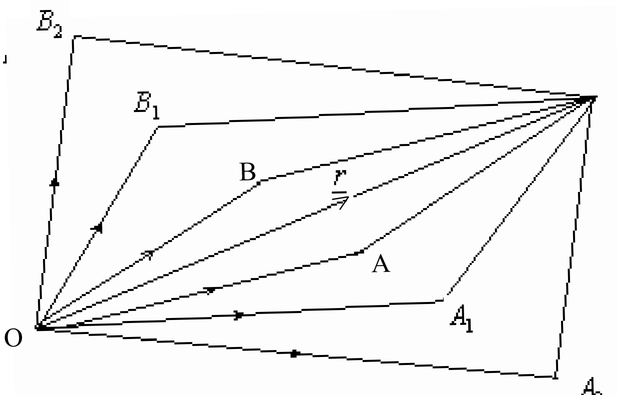
5.17 Resolution of a given Vector into two components

Let the given vector $\underline{r}$ be represented by $\overrightarrow{OP}$.

Complete the parallelogrm OAPB with OP as a diagonal. If the vectors represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are $\underline{a}$ and $\underline{b}$



from the parallelogram law of vector addition $\underline{a}+\underline{b}=\underline{r}$ That is $\underline{r}=\underline{a}+\underline{b}$. Hence the given vector $\underline{r}$ can be resolved into the two vectors $\underline{a}$ and $\underline{b}$ and $\underline{a}$ and $\underline{b}$ are the resolved parts of the vector $\underline{r}$. If the directions of $\overrightarrow{OA}$ and $\overrightarrow{OB}$ make angles α and β with $\overrightarrow{OP}$, as shown in the diagram, applying the sine rule to OAP,



$$\frac{OA}{\sin\beta}=\frac{AP}{\sin\alpha}=\frac{OP}{\sin[180-(\alpha+\beta)]}$$

$$\frac{|\underline{a}|}{\sin\beta}=\frac{|\underline{b}|}{\sin\alpha}=\frac{|\underline{r}|}{\sin(\alpha+\beta)}$$

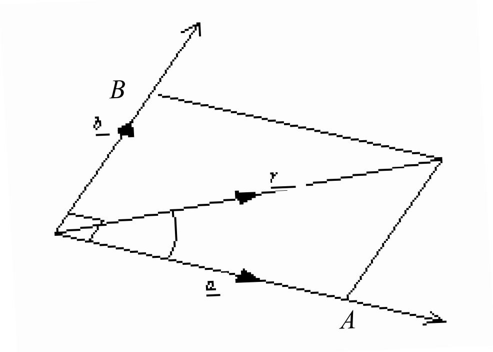
$$|\underline{a}| = \frac{|\underline{r}|\sin\beta}{\sin(\alpha+\beta)}, \quad |\underline{b}| = \frac{|\underline{r}|\sin\alpha}{\sin(\alpha+\beta)}$$

An infinite number of parallelogram can be drawn as shown above with OP as the diagonal. Hence the given vector $\underline{r}$ can be resolved into two vectors in an infinite number of ways.

5.18 Resolving a vector in two given perpendicular directions $\overrightarrow{OX}$ and $\overrightarrow{OY}$

Represent the vector by $\overrightarrow{OP}$ and by drawing lines parallel to OX and OY through O complete the parallelogram OAPB.

If the directions $\overrightarrow{OX}$ and $\overrightarrow{OY}$ make angles θ and $(90-\theta)$ respectively with OP, from OAPΔ



$$\cos\theta=\frac{OA}{OP},$$

$$OA=(OP)\cos\theta$$

$$\therefore\ |\underline{a}|=|\underline{r}|\cos\theta$$

$$\sin\theta=\frac{AP}{OP},\quad AP=(OP)\sin\theta \qquad \overrightarrow{AP}=\overrightarrow{OB}=\underline{b}$$

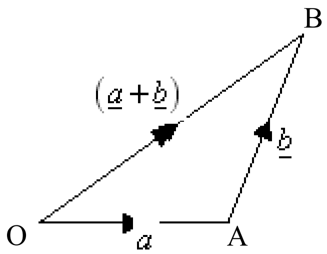
$$\therefore\ |\underline{b}|=|\underline{r}|\sin\theta$$

If the magnitudes of the vectors $\underline{r},\underline{a},\underline{b}$ are r,a,b respectively then $a=r\cos\theta,\ b=r\sin\theta$

5.19 Properties of Vector Addition

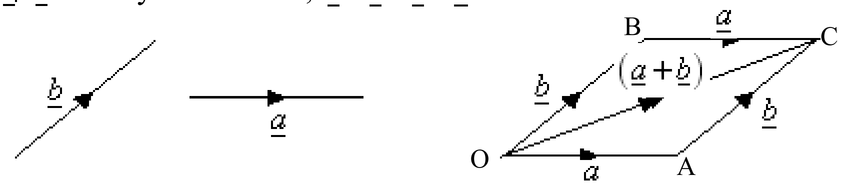
a. Closure property of vector addition

The result of adding two vectors is also a vector. That is if $\underline{a}$ and $\underline{b}$ are any two vectors $(\underline{a}+\underline{b})$ is also a vector. (This is evident from the vector law of addition).



***b.* Commutative law on vector addition**

If $\underline{a}, \underline{b}$ are any two vectors, $\underline{a}+\underline{b}=\underline{b}+\underline{a}$



Proof : Represent the two vectors $\underline{a}, \underline{b}$ by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively and complete the parallelogram OACB.

$BC = OA \quad BC \parallel OA$, sense from B to C

$\therefore \quad \overrightarrow{BC} = \overrightarrow{OA} = \underline{a}$ also $\overrightarrow{AC} = \overrightarrow{OB} = \underline{b}$

$\underline{a}+\underline{b} = \overrightarrow{OA}+\overrightarrow{AC} = \overrightarrow{OC}$ -------(1) (Triangle law)

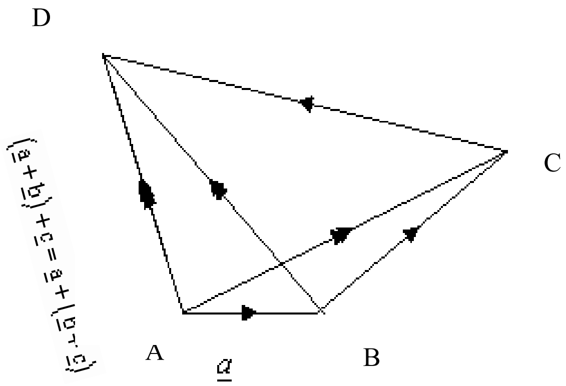
$\underline{b}+\underline{a} = \overrightarrow{OB}+\overrightarrow{BC} = \overrightarrow{OC}$------- (2) (Triangle law)

$\therefore \ (1)$ and (2) $\ \underline{a}+\underline{b} = \underline{b}+\underline{a}$

***c.* Associative law on vector addition**

If $\underline{a}, \underline{b}$ and $\underline{c}$ are any three vectors,

$(\underline{a}+\underline{b})+\underline{c} = \underline{a}+(\underline{b}+\underline{c})$



Proof : Represent the three vectors $\underline{a}, \underline{b}, \underline{c}$ by $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}$ respectively.

$(\underline{a}+\underline{b})+\underline{c} = (\overrightarrow{AB}+\overrightarrow{BC})+\overrightarrow{CD} = \overrightarrow{AC}+\overrightarrow{CD} = \overrightarrow{AD}$ -------(1) (Triangle law)

$\underline{a}+(\underline{b}+\underline{c}) = \overrightarrow{AB}+(\overrightarrow{BC}+\overrightarrow{CD}) = \overrightarrow{AB}+\overrightarrow{BD} = \overrightarrow{AD}$ .... (2) (Triangle law)

$\therefore$ From (1) and (2) $\ (\underline{a}+\underline{b})+\underline{c} = \underline{a}+(\underline{b}+\underline{c})$

As vector addition agrees with the commulative and associative laws in adding several vectors they can be added in any order.

***d.* Distributive law**

Let $\underline{a}$ and $\underline{b}$ be any two vectors and λ and μ be any two scalars.

Then

$(i)\ \lambda(\underline{a}+\underline{b}) = \lambda\underline{a}+\lambda\underline{b}$

$(ii)\ (\lambda+\mu)\underline{a} = \lambda\underline{a}+\mu\underline{b}$

Example:

If $|\underline{a}| = |\underline{b}|$ show that $(\underline{a}+\underline{b}) \perp (\underline{a}-\underline{b})$

Solution : Represent the vector $\underline{a}$ by $\overrightarrow{OA}$ and the vector $\underline{b}$ by $\overrightarrow{OB}$. As $|\underline{a}| = |\underline{b}|$, $OA = OB$, Complete the rhombus OACB. From the parallelogram law, $\underline{a}+\underline{b}=\overrightarrow{OC}$

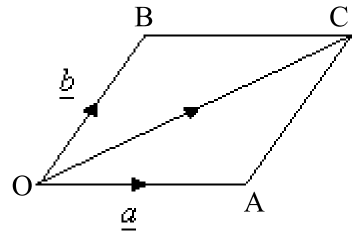
BC = OA, CA = BO

BC // OA, CA // BO.

Sense from B to C = Sense from O to A.

Sense from C to A = Sense from B to O.

AS OACB is a rhombus,



$\therefore\ \overrightarrow{BC}=\overrightarrow{OA}=\underline{a}$

$\therefore\ \overrightarrow{CA}=\overrightarrow{BO}=-\overrightarrow{OB}=-\underline{b}$

$\underline{a}-\underline{b}=a+(-\underline{b})=\overrightarrow{BC}+\overrightarrow{CA}=\overrightarrow{BA}.\quad \overrightarrow{BA}=\underline{a}-\underline{b}$

$OC\perp BA \quad \therefore\ (\underline{a}+\underline{b})\perp(\underline{a}-\underline{b})$

Example : denote in simplest form.

i. $\overrightarrow{PQ}+\overrightarrow{QR}+\overrightarrow{RS}+\overrightarrow{ST}$ ii. $\overrightarrow{AD}+\overrightarrow{AB}+\overrightarrow{DC}+\overrightarrow{BC}$ iii. $\overrightarrow{LM}-\overrightarrow{PN}+\overrightarrow{PQ}-\overrightarrow{NM}$

Solution :

i. $\overrightarrow{PQ}+\overrightarrow{QR}+\overrightarrow{RS}+\overrightarrow{ST} = \left(\overrightarrow{PQ}+\overrightarrow{QR}\right)+\left(\overrightarrow{RS}+\overrightarrow{ST}\right)$

$=\overrightarrow{PR}+\overrightarrow{RT}$ (From triangle law) $=\overrightarrow{PT}$ (From triangle law)

ii. $\overrightarrow{AD}+\overrightarrow{AB}+\overrightarrow{DC}+\overrightarrow{BC}$

$=\left(\overrightarrow{AD}+\overrightarrow{DC}\right)+\left(\overrightarrow{AB}+\overrightarrow{BC}\right)$ (As vectors can be added in any order)

$=\overrightarrow{AC}+\overrightarrow{AC}$ (From triangle law) $=2\,\overrightarrow{AC}$

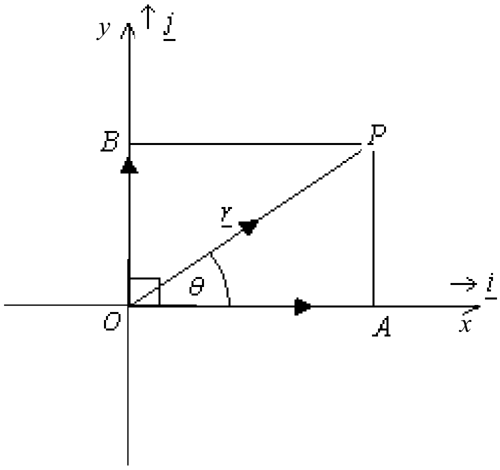
iii $\overrightarrow{LM}-\overrightarrow{PN}+\overrightarrow{PQ}-\overrightarrow{NM}$

$=\overrightarrow{LM}+\overrightarrow{NP}+\overrightarrow{PQ}+\overrightarrow{MN}$ (From the definition of a negative vector)

$=\left(\overrightarrow{LM}+\overrightarrow{MN}\right)+\left(\overrightarrow{NP}+\overrightarrow{PQ}\right)$ (As vector can be added in any order)

$=\overrightarrow{LN}+\overrightarrow{NQ}$ (From triangle law) $=\overrightarrow{LQ}$ (From triangle law)

5.20 Position Vector



The vector $\overrightarrow{OP}$ which denotes the position of the point relative to O is called the position vector of the point relative to O.

When several points are considered the position vector of each point is denoted relative to the same point. (origin)

5.21 Vectors in two dimensional space

If the co-ordinates of the point P relative to the rectangular system of axes Oxy are (x,y) $\overrightarrow{OP} = \underline{r} = x\underline{i} + y\underline{j}$ and $OP = \sqrt{x^2+y^2}$

Proof :

Let the co-ordinates of the point relative to the rectangular system of axes Oxy be (x,y)

$OA = x, OB = y$. The unit vectors along $\overrightarrow{OX}$ and $\overrightarrow{OY}$ are generally taken as $\underline{i}$ and $\underline{j}$ respectively.

$\overrightarrow{OA} = (OA)\underline{i} = x\underline{i}$, $\qquad \overrightarrow{OB} = (OB)\underline{j} = y\underline{j}$

$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{OB}$, $\qquad \underline{r} = x\underline{i} + y\underline{i}$

The vector $\underline{r}$ $\overrightarrow{OP} = \underline{r} = x\underline{i} + y\underline{j}$, can be resolved into the two vectors $x\underline{i}$ and $y\underline{j}$ along the directions $\overrightarrow{Ox}$ and $\overrightarrow{Oy}$ respectively.

If, $P\hat{O}X = \theta$ from $OAP\Delta$

$OA = (OP)\cos\theta$, $\quad OB = AP = (OP)\sin\theta$

$x = r\cos\theta$, $\quad y = r\sin\theta$ where $r = |\underline{r}|$

If $P \equiv (x,y)$ the position vector of P, $\overrightarrow{OP} = \underline{r} = x\underline{i} + y\underline{j}$

$OP = \sqrt{x^2+y^2}$ that is $|\underline{r}| = \sqrt{x^2+y^2}$ P

Example : $\underline{a} = x_1\underline{i} + y_1\underline{j}$ and $\underline{b} = x_2\underline{i} + y_2\underline{j}$ $\quad \underline{a} + \underline{b} \quad \underline{a} - \underline{b}$

Proof :

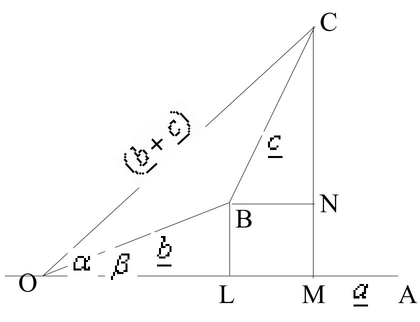
$$\underline{a} + \underline{b} = (x_1\underline{i} + y_1\underline{j}) + (x_2\underline{i} + y_2\underline{j})$$
$$= (x_1 + x_2)\underline{i} + (y_1 + y_2)\underline{j}$$

Similarly $\underline{a} - \underline{b} = (x_1 - x_2)\underline{i} + (y_1 - y_2)\underline{j}$

5.22 Vector Products

5.22.1 Scalar product (dot product)

If $\underline{a}$ and $\underline{b}$ are any two vectors, their scalar product is defined as $\underline{a}\cdot\underline{b} = |\underline{a}||\underline{b}|\cos\theta$ where θ is the angle between the directions of the vectors $\underline{a}$ and $\underline{b}$. As the result of this product is a scalar it is known as the scalar product. Further as this product is denoted by a dot it is also known as the dot product.



5.22.1.1 Properties of the Scalar product

i. Scalar product agrees with the commulative law. That is if $\underline{a}$ and $\underline{b}$ are any two vectors, $\underline{a}\cdot\underline{b} = \underline{b}\cdot\underline{a}$

Proof :

$$\underline{a}.\underline{b} = |\underline{a}||\underline{b}|\cos\theta = |\underline{a}||\underline{b}|\cos\theta = \underline{b}\cdot\underline{a}$$

$$\underline{a}\cdot\underline{b} = \underline{b}\cdot\underline{a}$$

ii. $(\underline{a}\cdot\underline{b})$ gives a scalar. The dot product with $\underline{c}$ cannot be considered with this scalar. Therefore $(\underline{a}\cdot\underline{b})\cdot\underline{c}$ is meaningless. Similarly $\underline{a}\cdot(\underline{b}\cdot\underline{c})$ is also meaningless. Therefore the dot product agree with the associative law is not discussed.

iii. Dot product is distributive.
That is if $\underline{a},\underline{b},\underline{c}$ are any three vectors, then
$\underline{a}\cdot(\underline{b}+\underline{c}) = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}$

Proof :

Let the vectors $\underline{a}\cdot\underline{b}$ and $\underline{c}$ be represented by $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{BC}$ respectively.

$\underline{a}\cdot(\underline{b}+\underline{c}) = \overrightarrow{OA}\cdot\left[\overrightarrow{OB}+\overrightarrow{BC}\right]$

When $\underline{a}\cdot\underline{b}$ and $\underline{c}$ are any three vectors,

$= \overrightarrow{OA}\cdot\overrightarrow{OC} = (OA)(OC)\cos\alpha$

$= (OA)(OM) ----(1)$

$\underline{a}\cdot\underline{b} = \overrightarrow{OA}\cdot\overrightarrow{OB} = (OA)(OB)\cos\beta = (OA)(OL)$

$\underline{a}\cdot\underline{c} = \overrightarrow{OA}\cdot\overrightarrow{BC} = (OA)(BC)\cos\theta = (OA)(BN) = (OA)(LM)$

$\underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c} = (OA)(OL)+(OA)(LM) = (OA)[OL+LM] = (OA)(OM) ---(2)$

From (1) and (2) we get, $\underline{a}\cdot(\underline{b}+\underline{c}) = \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}$

- When $\underline{a},\underline{b}$ and $\underline{c}$ are any three vectors then, $(\underline{a}+\underline{b})\cdot\underline{c} = \underline{a}\cdot\underline{c}+\underline{b}\cdot\underline{c}$

Proof:

$(\underline{a}+\underline{b})\cdot\underline{c} = \underline{c}\cdot(\underline{a}+\underline{b})$;dot product obeys commutative law

$= \underline{c}\cdot\underline{a}+\underline{c}\cdot\underline{b}$ from the above result

$(\underline{a}+\underline{b})\cdot\underline{c} = \underline{a}\cdot\underline{c}+\underline{b}\cdot\underline{c}$;dot product obeys commutative law

- when $\underline{a}$, $\underline{b}$, $\underline{c}$ and $\underline{d}$ re any three vectors then, $(\underline{a}+\underline{b})\cdot(\underline{c}+\underline{d}) = \underline{a}\cdot\underline{c}+\underline{a}\cdot\underline{d}+\underline{b}\cdot\underline{c}+\underline{b}\cdot\underline{d}$

Proof:

$(\underline{a}+\underline{b})\cdot(\underline{c}+\underline{d}) = \underset{\sim}{e}\cdot(\underline{c}+\underline{d})$ here $\underset{\sim}{e} = \underline{a}+\underline{b}$

$= \underset{\sim}{e} \cdot \underline{c} + \underset{\sim}{e} \cdot \underline{d}$ from the above result

$= (\underline{a}+\underline{b}) \cdot \underline{c} + (\underline{a}+\underline{b}) \cdot \underline{d}$ as $\underset{\sim}{e} = \underline{a}+\underline{b}$

$(\underline{a}+\underline{b}) \cdot (\underline{c}+\underline{d}) = \underline{a} \cdot \underline{c} + \underline{b} \cdot \underline{c} + \underline{a} \cdot \underline{d} + \underline{b} \cdot \underline{d}$

Similarly, when $\underline{a}, \underline{b}, \underline{c}, \underline{d}, \underline{e}$ and $\underline{f}$ are any six vectors then,

$(\underline{a}+\underline{b}+\underline{c}) \cdot (\underline{d}+\underset{\sim}{e}+\underline{f}) = \underline{a} \cdot \underline{d} + \underline{b} \cdot \underline{d} + \underline{c} \cdot \underline{d} + \underline{a} \cdot \underset{\sim}{e} + \underline{b} \cdot \underset{\sim}{e} + \underline{c} \cdot \underset{\sim}{e} + \underline{c} \cdot \underline{d} + \underline{c} \cdot \underset{\sim}{e} + \underline{c} \cdot \underline{f}$

In the same way the result can be extended to any number of vectors. When λ and μ are any two scalars and $\underline{a}, \underline{b}$ are any two vectors then, $\lambda \underline{a} \cdot \mu \underline{b} = \lambda\mu(\underline{a} \cdot \underline{b})$

- **Conditions for $\underline{a} \cdot \underline{b} = 0$**

$\underline{a} \cdot \underline{b} = 0 \quad \Leftrightarrow |\underline{a}| \times |\underline{b}| \times \cos\theta = 0 \quad \Leftrightarrow |\underline{a}| = 0$ or $|\underline{b}| = 0$ or $\cos\theta = 0$

$\Leftrightarrow \underline{a} = \underline{0}$ or $\underline{b} = \underline{0}$ or $\theta = 90^0 \Leftrightarrow \underline{a} = \underline{0}$ or $\underline{b} = \underline{0}$ or $\underline{a} \perp \underline{b}$

$\therefore \; \underline{a} \neq \underline{0}, \; \underline{b} \neq \underline{0}$ iy $\underline{a} \cdot \underline{b} = 0 \;\Rightarrow\; \underline{a} \perp \underline{b}$

$\underline{a} \cdot \underline{a} \;=\; |\underline{a}| \times |\underline{a}| \times \cos 0^\circ = |\underline{a}|^2, \quad (\because \cos 0^\circ = 1) \qquad \therefore \; |\underline{a}| = \sqrt{\underline{a} \cdot \underline{a}}$

Problem :

Use the scalar product to prove the folllowing :

i. The angle in a Semi circle is 90^0.

ii. In usual notation $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ for $ABC\Delta$

iii. Choosing suitable vectors $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

.

Solution :

Let the centre of the semicircle ABC be O. Also let the position vectors of A and C relative to O be $\underline{a}$ and $\underline{b}$ respectively.

Then $\overrightarrow{OA} = \underline{a}, \overrightarrow{OC} = \underline{c}$ and $OA = OC$, $|\underline{a}| = |\underline{c}|$

$OB = OA, \; OB // OA, O$

Sense from O to B = sense from A to O.

$\therefore \; \overrightarrow{OB} = -\overrightarrow{OA} = -\underline{a}$

$\overrightarrow{AC} = \underline{c} - \underline{a}, \quad \overrightarrow{BC} = \underline{c} - (-\underline{a}) = \underline{c} + \underline{a}$

$\therefore \overrightarrow{AC} \cdot \overrightarrow{BC} = (\underline{c} - \underline{a}) \cdot (\underline{c} + \underline{a}) = \underline{c} \cdot \underline{c} - \underline{a} \cdot \underline{c} + \underline{c} \cdot \underline{a} - \underline{a} \cdot \underline{a} = |\underline{c}|^2 - \underline{a} \cdot \underline{c} + \underline{a} \cdot \underline{c} - |\underline{a}|^2 = |\underline{c}|^2 - |\underline{a}|^2$

$\overrightarrow{AC} \cdot \overrightarrow{BC} = 0, \; |\underline{a}| = |\underline{c}|$,

$\overrightarrow{AC} \neq \underline{0}, \overrightarrow{BC} \neq \underline{0} \quad \therefore \overrightarrow{AC} \perp \overrightarrow{BC} \;\Rightarrow\; AC \perp BC \Rightarrow A\hat{C}B = \underline{\underline{90^0}}$

$\therefore \overrightarrow{AC} \perp \overrightarrow{BC} \;\Rightarrow\; AC \perp BC \Rightarrow A\hat{C}B = \underline{\underline{90^0}}$

ii. In the $\triangle ABC$ $\overrightarrow{CB}, \overrightarrow{CA}, \overrightarrow{AB}$ let $\underline{a}, \underline{b}, \underline{c}$ be denoted by $\overrightarrow{CA} + \overrightarrow{AB} = \overrightarrow{CB}$ respectively.

Then $BC = \left|\overrightarrow{CB}\right| = |\underline{a}| = a$ and $CA = \left|\overrightarrow{CA}\right| = |\underline{b}| = b$ and $AB = \left|\overrightarrow{AB}\right| = |\underline{c}| = c$

$\overrightarrow{CA} + \overrightarrow{AB} = \overrightarrow{CB}$ (law)

$$\underline{b} + \underline{c} = \underline{a}$$

$$(\underline{b} + \underline{c}) \cdot (\underline{b} + \underline{c}) = \underline{a}\underline{a}$$

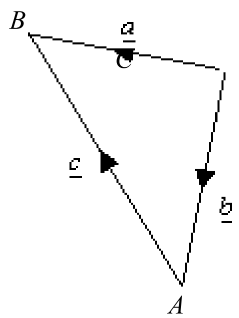
$$\underline{b} \cdot \underline{b} + \underline{b} \cdot \underline{c} + \underline{c} \cdot \underline{b} + \underline{c} \cdot \underline{c} = \underline{a} \cdot \underline{a}$$

$$|\underline{b}|^2 + \underline{c} \cdot \underline{b} + \underline{c} \cdot \underline{b} + |\underline{c}|^2 = |\underline{a}|^2$$

$$b^2 + 2\underline{c} \cdot \underline{b} + c^2 = a^2$$

$$b^2 + 2|\underline{c}| \cdot |\underline{b}| \cos(\pi - A) + c^2 = a^2 \Rightarrow b^2 + 2cb(-\cos A) + c^2 = a^2$$

$$b^2 + c^2 - a^2 = 2bc \cos A \qquad \Rightarrow \qquad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$



iii. $A(x_1, y_1), B(x_2, y_2)$

$$\therefore \underline{a} = \overrightarrow{OA} = x_1\underline{i} + y_1\underline{j} \quad \& \quad \underline{b} = \overrightarrow{OB} = x_2\underline{i} + y_2\underline{j}$$

Then $\underline{a} \cdot \underline{b} = \overrightarrow{OA} \cdot \overrightarrow{OB} = (OA)(OB) \times \cos(\alpha - \beta)$

That is $(x_1\underline{i} + y_1\underline{j}) \cdot (x_2\underline{i} + y_2\underline{j}) = |x_1\underline{i} + y_1\underline{j}||x_2\underline{i} + y_2\underline{j}| \cos(\alpha - \beta)$

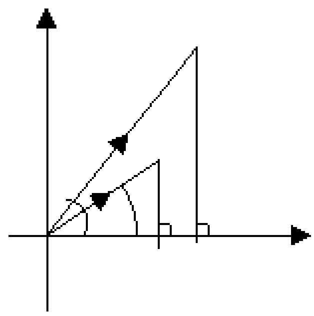
$$\Rightarrow x_1x_2 + y_1y_2 = \sqrt{x_1^2 + y_1^2} \times \sqrt{x_2^2 + y_2^2} \cos(\alpha - \beta)$$

$$\Rightarrow \cos(\alpha - \beta) = \frac{x_1}{\sqrt{x_1^2 + y_1^2}} \times \frac{x_2}{\sqrt{x_2^2 + y_2^2}} + \frac{y_1}{\sqrt{x_1^2 + y_1^2}} \cdot \frac{y_2}{\sqrt{x_2^2 + y_2^2}}$$

$$\Rightarrow \cos(\alpha - \beta) = \frac{x_1}{\sqrt{x_1^2 + y_1^2}} \times \frac{x_2}{\sqrt{x_2^2 + y_2^2}} + \frac{y_1}{\sqrt{x_1^2 + y_1^2}} \cdot \frac{y_2}{\sqrt{x_2^2 + y_2^2}}$$

$$\cos(\alpha - \beta) = \cos\alpha . \cos\beta + \sin\alpha . \sin\beta$$

5.22.2 Vector product of two vectors (Cross product)



The vector product of any two vectors $\underline{a}$ and $\underline{b}$ is defined as $\underline{a} \times \underline{b} = |\underline{a}| \times |\underline{b}| \times \sin\theta\, \underline{n}$ where θ is the angle between $\underline{a}$ and $\underline{b}$ and $\underline{n}$ is a unit vector perpendicular to both $\underline{a}$ and $\underline{b}$ and in the direction of the pointed end of a right handed screw turned from $\underline{a}$ to $\underline{b}$

$$\underline{b} \times \underline{a} = |\underline{b}| \times |\underline{a}| \times \sin\theta(-\underline{n}) = -|\underline{a}| \times |\underline{b}| \times \sin\theta(\underline{n})$$

$\therefore \underline{b} \times \underline{a} = -\underline{a} \times \underline{b}$ That is $\underline{b} \times \underline{a} \neq \underline{a} \times \underline{b}$

Vector product is not commutative.

But $\therefore$ $|\underline{b} \times \underline{a}| = |\underline{a} \times \underline{b}| = |\underline{a}| \times |\underline{b}| \sin\theta$

If $\underline{a} \times \underline{b}$ then, $\underline{a} \neq \underline{0}, \underline{b} \neq \underline{0}$ and if $\underline{a} \wedge \underline{b} = \underline{0}$ then

Proof :

$|\underline{a}|\times|\underline{b}|\times\sin\theta\,\underline{n}=\underline{0} \Rightarrow |\underline{a}|=0$ or $|\underline{b}|=0$ or $\sin\theta=0 \Rightarrow \underline{a}=\underline{0}$ or $\underline{b}=0$ or $\theta=0^0, 180^0$

$\Rightarrow \underline{a}=\underline{0}$ or $\underline{b}=\underline{0}$ or $\underline{a}\,//\,\underline{b}$ If $\underline{a}\neq\underline{0}, \underline{b}\neq\underline{0}$ and if $\underline{a}\wedge\underline{b}=\underline{0}$ then $\underline{a}\,//\,\underline{b}$

- **Vector Product is not associative**

Generally since $(\underline{a}\times\underline{b})\times\underline{c}\neq\underline{a}\times(\underline{b}\times\underline{c})$ vector procuct is not associative.

For any vectors $\underline{a},\underline{b},\underline{c}$

$\underline{a}\times(\underline{b}+\underline{c})=\underline{a}\times\underline{b}+\underline{a}\times\underline{c}$ and $(\underline{a}+\underline{b})\times\underline{c}=\underline{a}\times\underline{c}+\underline{b}\times\underline{c}$

Vector product is distributive. (Proof is necessary).

If $\underline{a}=a_1\underline{i}+a_2\underline{j}+a_3\underline{k}$ and $\underline{b}=b_1\underline{i}+b_2\underline{j}+b_3\underline{k}$

$$\underline{a}\times\underline{b}=(a_1\underline{i}+a_2\underline{j}+a_3\underline{k})\times(b_1\underline{i}+b_2\underline{j}+b_3\underline{k})$$

$$=a_1b_1\underline{i}\times\underline{i}+a_1b_2\underline{i}\times\underline{j}+a_1b_3\underline{i}\times\underline{k}\quad+a_2b_1\underline{j}\times\underline{i}+a_2b_2\underline{j}\times\underline{j}+a_2b_3\underline{j}\times\underline{k}$$

$$+a_3b_1\underline{k}\times\underline{i}+a_3b_2\underline{k}\times\underline{j}+a_3b_3\underline{k}\times\underline{k}$$

$$=\underline{0}+a_1b_2(\underline{k})+a_1b_3(-\underline{j})\quad+a_2b_1(-\underline{k})+\underline{0}+a_2b_3(\underline{i})\quad+a_3b_1(\underline{j})+a_3b_2(-\underline{i})+\underline{0}$$

$$=\underline{\underline{(a_2b_3-a_3b_2)\underline{i}-(a_1b_3-a_3b_1)\underline{j}+(a_1b_2-a_2b_1)\underline{k}}}$$

A convenient method to remember this result.

$$\underline{a}\times\underline{b}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}=\underline{i}\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix}-\underline{j}\begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix}+\underline{k}\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}$$

$$=\underline{i}(a_2b_3-a_3b_2)-\underline{j}(a_1b_3-a_3b_1)+\underline{k}(a_1b_2-a_2b_1)$$

Example :

If $\underline{a}=2\underline{i}-\underline{j}+2\underline{k}$ and $\underline{b}=\underline{i}-3\underline{k}$ then, $\underline{a}\times\underline{b}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 2 & -1 & 2\\ 1 & 0 & -3\end{vmatrix}$

$$=\underline{i}\begin{vmatrix}-1 & 2\\ 0 & -3\end{vmatrix}-\underline{j}\begin{vmatrix}2 & 2\\ 1 & -3\end{vmatrix}+\underline{k}\begin{vmatrix}2 & -1\\ 1 & 0\end{vmatrix}$$

Exercise

1. ABCDEF is a hexagon. If $\overrightarrow{AB} = \underline{a}$, $\overrightarrow{BC} = \underline{b}$ find $\overrightarrow{AC}, \overrightarrow{AD}, \overrightarrow{AF}, \overrightarrow{AE}, \overrightarrow{CE}$ in terms of $\underline{a}$ and $\underline{b}$

2. ABCDEF
   is a regular hexagon. Show that $\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{EA} + \overrightarrow{FA} = 4\overrightarrow{AB}$

3. ABCD is a parallelogram. E and F are the mid points of AD and CD respectively. Express the vectors $\overrightarrow{BE}$ and $\overrightarrow{BF}$ in terms of the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$ and show that
   $\overrightarrow{BE} + \overrightarrow{BF} = \frac{3}{2}\overrightarrow{BD}$

4. In $ABC\ \Delta$ D and E are the mid points of AB and AC respectively. Show that
   $\overrightarrow{BE} + \overrightarrow{DC} = \frac{2}{3}\overrightarrow{BC}$

5. D, E, F are the mid points of the sided of the triangle ABC. Show that
   $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OD} + \overrightarrow{OE} + \overrightarrow{OE}$

6. Show that the points with the following position vectors are rectilinear

   (i) $\underline{a},\ \underline{b},\ 3\underline{a} - 2\underline{b}$ $\quad$ (ii) $-2\underline{a} + 3\underline{b} + 5\underline{c},\ \underline{a} + 2\underline{b} + 3\underline{c},\ 7\underline{a} - \underline{c}$

   (iii) $\underline{i}+2\underline{j}+3\underline{k},\ -2\underline{i}+3\underline{j}+5\underline{k},\ 7\underline{i} - \underline{k}$ $\quad$ (iv) $\underline{a} - 2\underline{b} + 3\underline{c},\ 2\underline{a} + 3\underline{b} - 4\underline{c},\ 7\underline{b} + 10\underline{c}$

7. i. In the triangle ABC prove that $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \underline{0}$
   ii. In the quadrilateral ABCD prove that $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DA} = \underline{0}$
   iii. In the pentagon ABCDE prove that $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EA} = \underline{0}$

8. i. The mid points of the sides BC, CA, AB of the triangle ABC are D, E and F., Prove that $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OD} + \overrightarrow{OE} + \overrightarrow{OF}$
   ii. The mid points of the sides of the quadrilateral ABCD are ;P, Q, R, S. Prove that
   $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} = \overrightarrow{OP} + \overrightarrow{OQ} + \overrightarrow{OR} + \overrightarrow{OS}$

9. i. The mid points of the line segments AB and A'B' are E and G. Prove that
   $\overrightarrow{AA'} + \overrightarrow{BB'} = 2\overrightarrow{GG'}$
   ii. The mid points of the sides AB and BC of the triangle ABC are E and D. Prove that
   $\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{BC} = 4\overrightarrow{ED}$

10. If $\overrightarrow{OA} = \underline{a}$, $\overrightarrow{AC} = \underline{b}$. Express the vector $\overrightarrow{OC}$ of the parallelogram OACB in terms of $\underline{a}$ and $\underline{b}$.

(i) $|\underline{a}+\underline{b}|=|\underline{a}-\underline{b}|$ find the angle between $\underline{a}$ and $\underline{b}$

(ii) $|\underline{a}|=|\underline{b}|=|\underline{a}-\underline{b}|$ find the angle between $\underline{a}$ and $\underline{b}$

11 (i) $\underline{a} = \underline{i}-2\underline{j}$, $\underline{b} = 4\underline{i}$, $\underline{c} = 3\underline{i}-\underline{j}$ find $\underline{a}+\underline{b}+\underline{c}$ Find the unit vector along $\underline{a}+\underline{b}+\underline{c}$

(ii) O is the origin and A,B, and C are any three points on the Oxy plane and if $\overline{OA} = 2\underline{j}$, $\overline{OB} = -\underline{i}+5\underline{j}$, $\overline{OC} = 2\underline{i}+4\underline{j}$ find $\overline{AB}, \overline{BC}, \overline{CA}$ and hence show that ABC is an isosceles triangle.

12. $\underline{b}=4\underline{a}$ and $\underline{c}=-2\underline{b}$ are two relations of the vectors $\underline{a},\underline{b},\underline{c}$ which are not null vectors.

Find i. $\dfrac{|\underline{a}|}{|\underline{b}|}$ ii. angle between $\underline{a}$ and $\underline{b}$. iii, angle between $\underline{b}$ and $\underline{c}$

13. $\overline{OA} = \underline{i}+2\underline{j}$, $\overline{OB}=3\underline{i}-\underline{j}$, $\overline{OC}=\underline{i}$

i. $|\overline{OA}+\overline{OB}|$ ii. $|\overline{AB}+\overline{BC}|$ iii. $|\overline{AB}-\overline{AC}|$

14. If $\overline{OA}=\underline{i}+2\underline{j}$, $\overline{OB}=3\underline{i}-\underline{j}$, $\overline{OC}=\underline{i}$ $\overline{AB}$ and $\overline{CA}$. Find AB and CA. Hence show that the points A, B, and C are rectilinear.

15. The points A, B and C are such that $\overline{OA}=2\underline{i}+3\underline{j}$, $\overline{OB}=6\underline{i}+6\underline{j}$, $\overline{OC}=\underline{i}$ where O is the origin. Find $\overline{AB}, \overline{BC}, \overline{CA}$ Hence find the lengths of the sides of the triangle ABC.

16. Prove the following using the scalar productg.
- i. The diagonals of the rhombus are perpendicular to each other.
- ii. Pythagoras Theorem
- iii. Pairs of opposite sides of a tetrahedron, are perpendicular to each other.
- iv. The attitudes of a tringle are concurrent.
- v. By selecting suitable vectors $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

17. If $\underline{a}=2\underline{i}-\underline{j}+\underline{k}$, $\underline{b}=-3\underline{i}+\underline{j}+\underline{k}$ and $\underline{c}=\underline{i}-2\underline{j}+3\underline{k}$ find the following vectors.

i. $\underline{b}+\underline{c}$ ii. $\underline{a}\times(\underline{b}+\underline{c})$ iii. $\underline{a}\times\underline{b}$ iv. $\underline{a}\times\underline{c}$

v. $\underline{a}\times(\underline{b}+\underline{c})=\underline{a}\times\underline{b}+\underline{a}\times\underline{c}$

18. For $\underline{a},\underline{b},\underline{c}$ in the above problem, find, i. $(\underline{a}\times\underline{b})\times\underline{c}$ ii. $\underline{b}\times\underline{c}$ iii. $\underline{a}\times(\underline{b}\times\underline{c})$

iv. See whether $(\underline{a}\times\underline{b})\times\underline{c}$ and $\underline{a}\times(\underline{b}\times\underline{c})$ are equal.

6. Forces Acting on a Particle

Competency : 2 Interprets systems of co-planer forces to apply instances of the existence or non-existence of equilibrium in day to day life

Competency Level : 2.1, 2.2 and 2.3
Pages 26, 27, 28, 29 of the Grade 12 Teacher's Instructional Manual

Subject Content : Equilibrium of a particle

By studying this section you will develop the skills of

- describing the concept of a particle.
- describing the concept of a force.
- identifying the various types of forces in mechanics.
- describing the resultant of a system of forces.
- expressing the parallelogram law of forces.
- finding the resultant of two forces graphically using the parallelogram law of forces.
- deriving formulae using the parallelogram law of forces.
- solving problems using the parallelogram law of forces.
- identifying the components of a force.
- resolving a force into two given directions.
- resolving a force into two directions perpendicular to each other.
- expressing the polygon law of forces which is used to find the resultant of a system of forces acting at a point.
-
- explaining the equilibrium of a system of forces.
- expressing the conditions under which a particle is kept in equilibrium by the action of two forces.
- expressing the conditions under which a particle is kept in equilibrium by the action of Three co-planer forces.
- expressing and proving the triangle law of forces, its converse and Lami's theorem.
- expressing the conditions under which a system of forces acting on a particle is in equilibrium.
- solving problems on the equilibrium of a system of forces acting on a particle.

Introduction

The students have studied about forces, the units of force, the parallelogram law of forces and the resolution of forces under the grades 9 -11 science syllabus. Here, systems of co-planer forces acting on a particle, which come under the combined mathematics section on statics will be studied. How the resultant of a system of co-planer forces acting on a particle is found and the equilibrium of forces will be discussed. The application of the converse of the triangle law of forces and the use of geometry and trigonometry in solving problems related to the equilibrium of three co-planer forces will also be discussed here.

6.1 Particle

- ♦ A particle is a body which has dimensions which are negligible in comparison to other bodies under consideration.
- ♦ Geometrically a particle is represented by a point.
- ♦ Therefore, forces acting on a particle are considered to be concurrent.

6.2 Force

- ♦ A force is an influence on a body which is at rest which causes it to move, or on a moving body which causes a change in the nature of its motion.
- ♦ A force has a magnitude, direction and line of action, and thus is a vector.
- ♦ The SI unit of force is Newton (N). A force is also measured in kilogramme weight (kgwt), gramme weight (gwt) etc.
- ♦ The forces in mechanics can be divided into three main types.

  i. Forces of Attraction
  - An attraction is a force exerted by one body on another without the intervention of any visible instrument and without the bodies being necessarily in contact.
    Example: Gravitational Force

  ii. Stresses (Tensions or Thrusts)
  - Stresses are forces which act along light inextensible strings or rods.
  - There are two types of stresses, namely tensions and thrusts.
  - The stress in a light inextensible string is a tension.
  - The stress in a light rod could be either a tension or a thrust.

  iii. Reactions
  - When two bodies are in contact with each other, the forces acting at the point of contact are called reactions.

Example:
When a body is placed on a smooth horizontal plane, the action on the body by the plane and vice versa.
When a particle is placed on a rough inclined plane, the action by the particle on the plane and vise versa.

6.3 The Resultant of a System of Forces Acting on a Particle

When several forces act on a body at the same instance, if a single force can be found whose effect on the body is the same as that of all the other forces taken together, then this single force is called the resultant of the other forces.

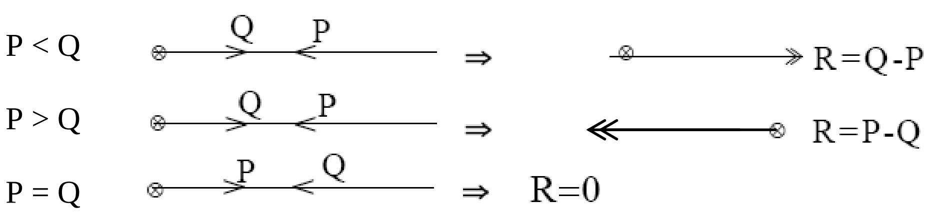
6.3.1 The Resultant of Two Forces acting on a Particle

i. When the two forces act in the same straight line and in the same direction

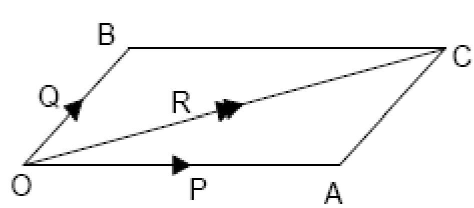
If the resultant is R, of two forces P and Q acting on a particle in the same straight line and in the same direction, then R = P + Q, and the direction of R is the same as the direction of P and Q.

ii. When the two forces act in the same straight line but in opposite directions

If P < Q, then R = Q – P, and the direction of R is the same as the direction of Q.
If P > Q, then R = P – Q, and the direction of R is the same as the direction of P.
If P = Q, then R = P – P = 0. The resultant in this case is zero.



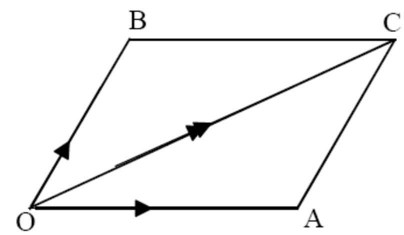
iii. When determining the resultant of two forces acting on a particle in non-parallel, non-collinear directions, the parallelogram of forces is used.



6.3.2 The Parallelogram Law of Forces

If two non-parallel non-collinear forces acting on a particle are represented in magnitude and direction by the two adjacent sides of a parallelogram, their resultant is represented both in magnitude and direction by the diagonal of the parallelogram passing through the vertex which is the intersection point of the relevant adjacent sides.

i.e., if two forces P and Q (which are neither collinear nor parallel), acting on a particle at point O, are represented in magnitude and direction by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively, and the parallelogram OACB is constructed, the resultant R of the two forces P and Q is given by $\overrightarrow{OC}$.



6.3.3 Determining the Magnitude and the Direction of the Resultant

i. Graphical Method ii. Using Formulae

i. Graphical Method

Let us represent the force P by $\overrightarrow{OA}$ and the force Q by $\overrightarrow{OB}$ according to some scale, and construct the parallelogram OACB. Then the resultant R is represented by $\overrightarrow{OC}$. Therefore, by measuring the length of OC and applying the scale, the magnitude of the resultant force R can be determined. By measuring the angle $A\hat{O}C$, the angle which the resultant makes with the force P can be found. Thereby the direction of the resultant is obtained.

Example: Two forces equal to 8N and 6N act on a particle. The angle between their lines of action is 60^o. Determine the magnitude and the direction of the resultant, graphically.

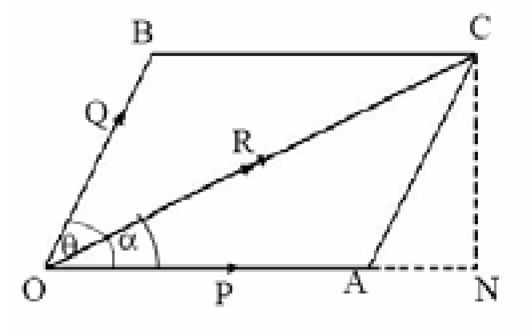
Solution

* First make a sketch.
* To represent a force by a line segment a suitable scale should initially be selected.
* Let us take the scale : 1cm represents 2N.
* Draw a line segment OA such that OA = 4cm.
* Mark B such that $A\hat{O}B = 60^o$ and OB = 3cm.
* Since OACB should be a parallelogram, BC = OA = 4cm, AC = OB = 3cm. Therefore, the point of intersection of the arcs of length 4cm drawn from B and of length 3cm drawn from A is C.
* Since AC = 6.1cm, according to the scale, R = 12.2N. Since $A\hat{O}C = 25^o$, the resultant is 12.2N acting in a direction which makes an angle 25^o with the direction of the force 8N.

- Suppose R is the resultant of two forces P and Q acting on a particle at an angle θ ($\neq 0$), and α is the angle between R and P. If the magnitude of any three of P, Q, R, θ and α are given, then the other two quantities can be determined.

ii. Using the formulae

Using the principle of the Parallelogram of Forces, let us derive formulae to determine the magnitude and direction of the resultant.



Let us take the foot of the perpendicular dropped from C to OA or OA extended, as N.

$AC = OB = Q$; $C\hat{A}N = A\hat{O}B = \theta$.

From the triangle CAN,

$AN = (AC)\cos C\hat{A}N$; $\therefore$ $AN = Q\cos\theta$.

$CN = (AC)\sin C\hat{A}N$; $\therefore$ $CN = Q\sin\theta$.

Since OCN is a right angled triangle, by Pythagoras' Theorem,

$OC^2 = ON^2 + CN^2$.

$OC^2 = (OA + AN)^2 + CN^2 \Rightarrow R^2 = (P + Q\cos\theta)^2 + (Q\sin\theta)^2$.

$R^2 = P^2 + 2PQ\cos\theta + Q^2(\cos^2\theta + \sin^2\theta)$.

$\Rightarrow R^2 = P^2 + 2PQ\cos\theta + Q^2$

$\Rightarrow R^2 = P^2 + Q^2 + 2PQ\cos\theta$.

Let us now obtain an expression for the angle α that the resultant R makes with P.

$C\hat{O}N = \alpha$.

From Δ OCN, $\tan C\hat{O}N = \dfrac{CN}{ON} \Rightarrow \tan\alpha = \dfrac{Q\sin\theta}{P + Q\cos\theta}$.

$$\therefore \alpha = \tan^{-1}\left(\frac{Q\sin\theta}{P + Q\cos\theta}\right).$$

Also, if the angle that the resultant R makes with the force Q is β, by interchanging P and Q in the above formula we obtain $\beta = \tan^{-1}\left(\dfrac{P\sin\theta}{Q + P\cos\theta}\right)$.

In special cases, instead of using the above derived formulae, the knowledge of geometry and trigonometry can be used to find the resultant.

If the forces P and Q are perpendicular, then A and N coincide. Therefore, by Pythagoras' Theorem,

$OC^2 = OA^2 + AC^2 \Rightarrow R^2 = P^2 + Q^2$.

By substituting $\theta = \frac{\pi}{2}$ into the above derived formula too we obtain this result.

$$\tan\alpha = \frac{AC}{OA} = \frac{Q}{P} \quad \Rightarrow \quad \alpha = \tan^{-1}\left(\frac{Q}{P}\right).$$

By substituting $\theta = \frac{\pi}{2}$ into the previously derived formula for α too we obtain this result.

When P = Q, show that $R = 2P\cos\frac{\theta}{2}$ and that $\alpha = \frac{\theta}{2}$.

6.3.4 The Polygon Law of Forces to Determine the Resultant of a System of Co-planer Forces

If n co-planer forces P_1, P_2,…….P_n which act at a point O are represented by $\overrightarrow{OA_1}, \overrightarrow{A_1A_2}, \ldots\ldots, \overrightarrow{A_{n-1}A_n}$ respectively, the resultant is given by $\overrightarrow{OA_n}$.
(This can be proved by repeatedly applying the triangle law)

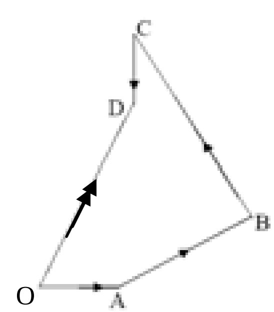
Note: The resultant of a system of forces acting on a particle can easily be found graphically by using the polygon principle.

Example: Find the magnitude and the direction of the resultant of the system of forces 2N, 4N, 6N, 2N acting on a point O. The angles between the first and second, the second and third and the third and fourth forces are respectively 30^o, 90^o and 150^o.

Solution:

Let us construct the polygon of forces for this system of forces.

Scale: 1cm represents 2N



If the straight line segments OA, AB, BC, CD are constructed such that the magnitude and direction of the forces 2N, 4N, 6N and 2N are represented by $\overrightarrow{OA}, \overrightarrow{AB}, \overrightarrow{BC}$ and $\overrightarrow{CD}$ respectively, then the magnitude and direction of the resultant is obtained from OD. By measuring, we obtaining OD = 2.8cm and $D\hat{O}A = 65^o$. Therefore, the resultant is 5.6 N and it makes an angle of 65^o with the first force.

Note: The resultant of more than two forces (system of forces) acting on a particle can be found by applying the Parallelogram Law of Forces repeatedly. However, the resultant of a system of forces acting on a particle can be found more easily by the resolution of forces.

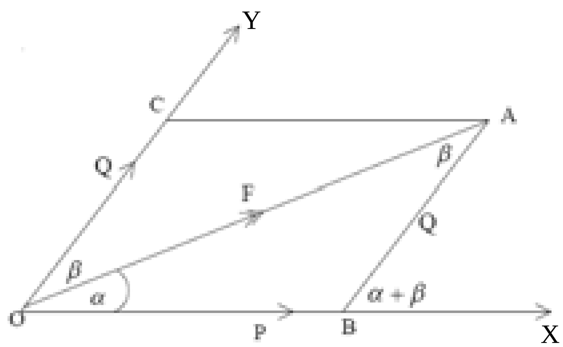
Let us now study about the resolution of forces.

6 .4 Resolution of Forces

Determining two or more forces that can be applied on a particle or at a point on a rigid body instead of a single force is called resolution of forces. The forces thus found are called the components of the original force.

Let us first consider resolving a force acting on a particle, in two given directions.

Let us resolve the force F acting at the point O, in the given two directions OX and OY. Let us assume that OX makes an angle α with the direction of the force F, and that OY makes an angle β with the direction of F, on the side opposite to that of OX.



Let us represent the force F by $\overrightarrow{OA}$. Now, by drawing two lines through A parallel to OX and OY, let us construct the parallelogram OBAC.

According to the Parallelogram Law of Forces, the resultant of the forces $\overrightarrow{OB}$ and $\overrightarrow{OC}$ is F. Therefore, the two forces which are represented by $\overrightarrow{OB}$ and $\overrightarrow{OC}$ can be applied on the particle instead of the force F.

That is, the given force F can be resolved into the two forces represented by $\overrightarrow{OB}$ and $\overrightarrow{OC}$ in the directions OX and OY respectively.

$\overrightarrow{OB}$ and $\overrightarrow{OC}$ are respectively the components of the force F in the directions OX and OY. Let us consider them to be P and Q respectively.

By applying the Sine Formula to the triangle OAB we obtain

$$\frac{OB}{\sin O\hat{A}B} = \frac{AB}{\sin A\hat{O}B} = \frac{OA}{\sin O\hat{B}A} \quad \Rightarrow \quad \frac{P}{\sin\beta} = \frac{Q}{\sin\alpha} = \frac{F}{\sin(\pi - (\alpha+\beta))}$$

$$\Rightarrow P = \frac{F\sin\beta}{\sin(\alpha+\beta)}, \qquad Q = \frac{F\sin\alpha}{\sin(\alpha+\beta)}$$

In this manner, a force F acting on a particle can be resolved in any two directions.

Example: Find the components of a 25N force acting on a particle, in the directions making angles of 30^o and 45^o with the direction of the force.

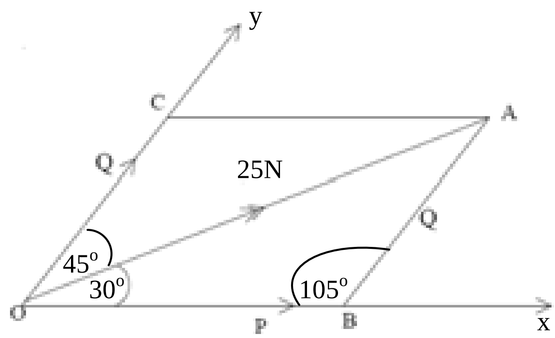
Solution:

Let us represent the force 25N by $\overrightarrow{OA}$. Let us represent the directions that make angles of 30^o and 45^o with OA by Ox and Oy respectively. Now let us construct the parallelogram OBAC as in the figure.

By applying the formulae

$$P = \frac{F\sin\beta}{\sin(\alpha+\beta)}, \qquad Q = \frac{F\sin\alpha}{\sin(\alpha+\beta)}$$

derived above we obtain,



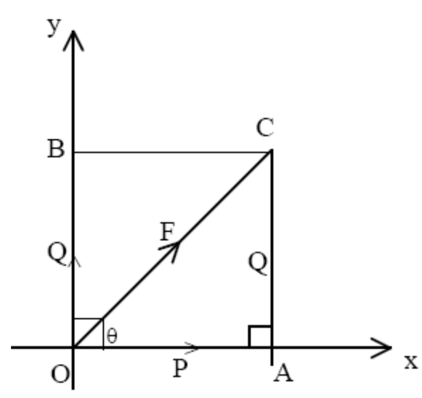
$$P = \frac{25\times\frac{1}{\sqrt{2}}}{\frac{\left(\sqrt{3}+1\right)}{2\sqrt{2}}} = \frac{50}{\sqrt{3}+1}$$

$$Q = \frac{25\times\frac{1}{2}}{\frac{\left(\sqrt{3}+1\right)}{2\sqrt{2}}} = \frac{25\sqrt{2}}{\left(\sqrt{3}+1\right)}$$

Therefore, the components of a force of 10N in the directions making angles of 30^o and 45^o with the direction of the force are respectively $\frac{20}{(\sqrt{3}+1)}$ N and $\frac{10\sqrt{2}}{(\sqrt{3}+1)}$ N.

Note: In solving various problems, the most important case of resolution is the resolution of forces in two perpendicular directions.

Let us resolve a given force F acting on a particle in two perpendicular directions. Let us assume that Ox makes an angle θ with the direction of the force F. Let us represent the force F in magnitude and direction by OC. If the two components are P and Q respectively, the parallelogram of forces is OACB, as given in the figure.



Then, $\frac{OA}{OC} = \cos C\hat{O}A \Rightarrow \frac{P}{F} = \cos\theta \Rightarrow P = F\cos\theta$

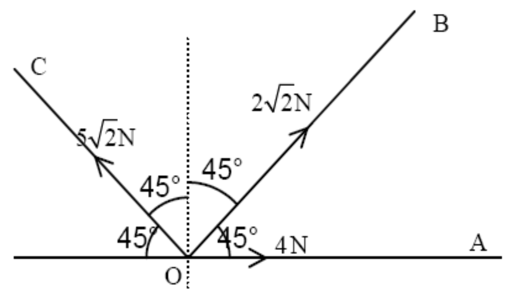
$\frac{CA}{OC} = \sin C\hat{O}A \Rightarrow \frac{Q}{F} = \sin\theta \Rightarrow Q = F\sin\theta$

Note: This result can also be obtained by taking $\alpha+\beta = \frac{\pi}{2}$ in the previously derived results.

Problem:

Three co-planer forces 4N, $2\sqrt{2}$ N and $5\sqrt{2}$ N act on a particle O in the directions OA, OB and OC respectively. If $A\hat{O}B = 45^{\circ}, B\hat{O}C = 90^{\circ}$, find the algebraic sum of the components of the forces in the direction of OA. Find also the algebraic sum of the components of the forces in the direction perpendicular to OA.

Solution:



The algebraic sum of the components in the direction of OA:

$$\rightarrow X = 4N + 2\sqrt{2}\cos 45^{\circ} N - 5\sqrt{2}\cos 45^{\circ} N$$

$$= 4N + 2\sqrt{2} \times \frac{1}{\sqrt{2}} N - 5\sqrt{2} \times \frac{1}{\sqrt{2}} N$$

$$= 4N + 2N - 5N = \underline{\underline{1N}}$$

The algebraic sum of the components in the direction perpendicular to OA:

$$\uparrow Y = 2\sqrt{2}\sin 45^{\circ} N + 5\sqrt{2}\sin 45^{\circ} N$$

$$= 2\sqrt{2} \times \frac{1}{\sqrt{2}} N + 5\sqrt{2} \times \frac{1}{\sqrt{2}} N$$

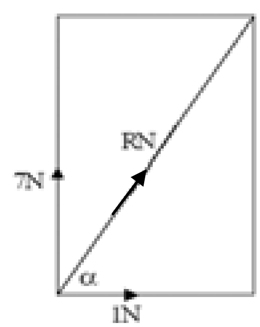
$$= 2N + 5N = \underline{\underline{7N}}$$

The resultant of a system of forces can be found by determining the components of the system in two directions perpendicular to each other.

Example:

Find the resultant of the system of forces given in the above problem.

Solution:



The above obtained system of forces X, Y is equivalent to the given system of forces. $\therefore$ the resultant of the original system of forces is the resultant of X and Y. X = 1N and Y = 7N.

Therefore, if R is the resultant,

$R = \sqrt{1^2 + 7^2} = \sqrt{1+49} = \sqrt{50} = 5\sqrt{2}$ N .

$\tan\alpha = \frac{7}{1} \Rightarrow \alpha = \tan^{-1} 7$.

The resultant of the given system of forces is of magnitude $5\sqrt{2}$ N, and it makes an angle $\alpha = \tan^{-1} 7$ with OA.

6.5 The Equilibrium of a System of Forces Acting on a Particle

6.5.1 The Equilibrium of Two Forces acting on a Particle

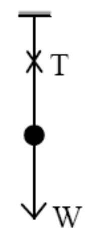
If a particle is in equilibrium under the action of two forces acting on it, then the two forces are equal in magnitude and opposite in direction.

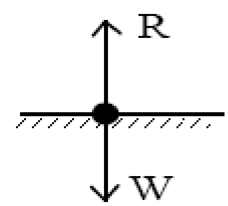
Conversely, if two forces of equal magnitude act in opposite directions on a particle, then the particle will be in equilibrium.

That is, the necessary and sufficient condition for two forces acting on a particle to be in equilibrium is that they should be equal in magnitude and opposite in direction.

Example:

i. If a heavy particle is suspended by a light inextensible string, its weight W acts vertically downwards. Let the tension in the string be T. If the particle is in equilibrium, then T = W and T and W act in opposite directions.





ii. When a heavy particle is placed on a horizontal plane, its weight W acts vertically downwards. The reaction of the plane acting on the particle is R. Since the particle is in equilibrium, R = W and R and W act in opposite directions.

6.5.2 The Equilibrium of Three Co-planer Forces Acting on a Particle

Prove the following results.

1. Three co-planer forces acting on a particle are in equilibrium if and only if the resultant of any two of the forces is equal in magnitude and opposite in direction to the third force.

2. If the three forces $\underline{P}$, $\underline{Q}$, $\underline{R}$ acting on a particle are in equilibrium, then they are co-planer.

3. If the three co-planer forces $\underline{P}$, $\underline{Q}$, $\underline{R}$ acting on a particle are in equilibrium, then the sum of the components in any direction is equal to zero.

4. **Triangle Law of Forces**: If three forces acting on a particle can be represented in magnitude and direction by the sides of a triangle, taken in order, then the three forces are in equilibrium.

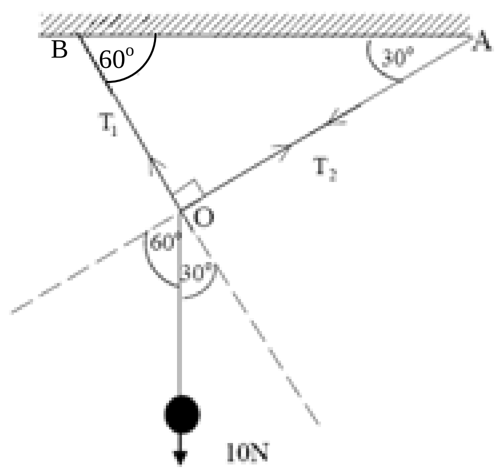
5. **The Converse of the Triangle Law of Forces**: If three forces acting on a particle are in equilibrium, then they can be represented in magnitude and direction by the three sides of a triangle taken in order.

6. **Lami's Theorem:** If a particle is in equilibrium under the action of three co-planer forces acting on it, each force is proportional to the sine of the angle between the other two forces.

Example:

A particle of weight 10N suspended by two light inextensible strings is in equilibrium in a vertical plane. If the two strings make angles of 30^o and 60^o with the downward vertical, find the tensions in the strings.
Solve this problem by resolving the forces, by the Triangle of Forces and by Lami's Theorem.

By the resolution of forces



Since the tensions T_1 and T_2 in the strings have to be determined, and since they are perpendicular to each other, the problem can easily be solved by resolving the forces in the directions of the strings.

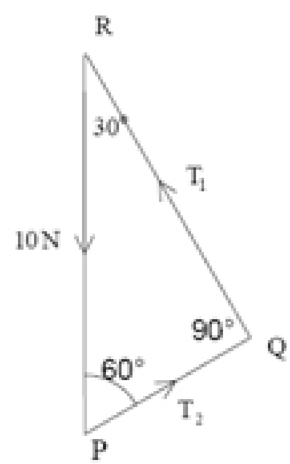
$$\nearrow \quad T_2 - 10\cos 60^0 = 0 \text{-----} \quad (1)$$

$$\nwarrow \quad T_1 - 10\sin 60^0 = 0 \text{-----} \quad (2)$$

From (1), $T_2 = 10\cos 60^0 = 5\text{N}$

From (2), $T_1 = 10\sin 60^0 = 5\sqrt{3}$

Using the Triangle of Forces
Let us construct the triangle of forces of the given system of forces.



In the triangle PQR,

$$\frac{RQ}{PR} = \cos 30^0 \quad \therefore \frac{T_1}{10} = \cos 30^0 \Rightarrow T_1 = 10\cos 30^0 \Rightarrow T_1 = 5\sqrt{3}\text{N}$$

$$\frac{PQ}{PR} = \sin 30^0, \quad \therefore \frac{T_2}{10} = \sin 30° \Rightarrow T_2 = 10\sin 30° \Rightarrow T_2 = 5\text{N}$$

By Lami's Theorem

$$\frac{10}{\sin 90^\circ} = \frac{T_1}{\sin 120^\circ} = \frac{T_2}{\sin 150^\circ}$$

$$\Rightarrow \quad \frac{10}{\sin 90^\circ} = \frac{T_1}{\sin 120^\circ} \quad \Rightarrow \quad T_1 = 10 \sin 120^\circ = 5\sqrt{3}\text{N}$$

$$\frac{10}{\sin 90^\circ} = \frac{T_1}{\sin 120^\circ} = \frac{T_2}{\sin 150^\circ}$$

$$\Rightarrow \quad \frac{10}{\sin 90^\circ} = \frac{T_2}{\sin 150^\circ} \quad \Rightarrow \quad T_2 = 10 \sin 150^\circ = 5\text{N}$$

Therefore the tensions in the strings are $5\sqrt{3}$ N and 5N.

When solving problems, the easiest and most suitable method should be used.

6.5.3 The Equilibrium of a System of Co-planer Forces Acting on a Particle

If the resultant of a system of co-planer forces acting on a particle is zero, then the system is in equilibrium.

The converse of this is also true.

That is, if a system of co-planer forces acting on a particle is in equilibrium, then the resultant is zero.

In the above situation, the following results can be used.

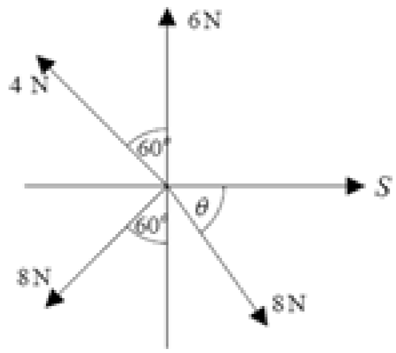
i. A polygon can be constructed with the sides of the polygon being equal in magnitude and in direction to the system of forces acting on the particle. This is a graphical method.

ii. If a system of co-planer forces acting on a particle is in equilibrium, then, the algebraic sum of the components in any direction is zero. This is a result used in calculations

Example:

If the system of forces in the figure is in equilibrium find S and θ

i. by the graphical method. ii. by calculation.



Solution

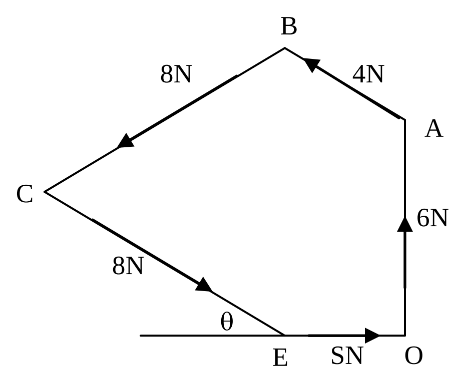
i. By the graphical method

Let us take the scale as 2N = 1cm.
The lines OA, AB and BC have been drawn such that OA is parallel and proportional to the force 6N, AB is parallel and proportional to the force 4N and BC is parallel and proportional to the force 8N. The line from C proportional to the force 8N, intersects the line through O parallel to S at E.

Then according to the Polygon Law of Forces

S = 2 |EO| = 2(1.73)

Therefore, S = 3.46N and $\theta = 30^o$.



ii. By calculation

If a system of forces is in equilibrium, then the sum of all the components in any direction is zero.

$$\rightarrow \quad S + 8\cos\theta - 4\sin 60^\circ - 8\sin 60^\circ = 0 - - - (1)$$

$$\uparrow \quad 6 + 4\cos 60^\circ - 8\cos 60^\circ - 8\sin\theta = 0$$

$$\therefore \quad 6 + 2 - 4 = 8\sin\theta \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = 30^\circ$$

$$(1) \Rightarrow S + 8\cos 30^o - 4\sin 60^o - 8\sin 60^o = 0 \Rightarrow S + 4\sqrt{3} - 6\sqrt{3} = 0 \Rightarrow S = 2\sqrt{3}$$

Exercises

1. P and Q are two forces acting at an angle θ. Their resultant is R. α is the angle that R makes with P. Find the following graphically.
   i. If P = 18 kg and Q = 10 kg, then R and α.
   ii. If P = 4.5 kg, Q = 5 kg and θ = 120°, then R and α.

2. Two forces P and Q act at a point O at an angle θ. Calculate the resultant of the two forces and the angle it makes with P.

i.	P = 6N,	Q = 12N,	$\theta = \frac{\pi}{2}$	ii.	P = 5N,	Q = 8N,	$\theta = \frac{\pi}{2}$
iii	P = 6N,	Q = 8N,	$\theta = 60^o$	iv.	P = 7N,	Q = 7N,	$\theta = 60^o$
v.	P = 5N,	Q = 6N,	$\theta = 60^o$	vi.	P = 8N,	Q = 5N,	$\theta = 120^o$

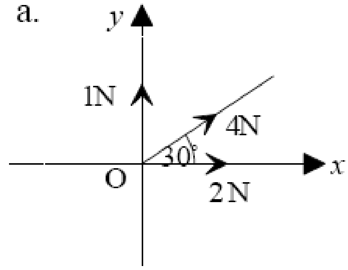
3. If R is the resultant of two forces P and Q acting on a particle at an angle θ (≠ 0), and if R makes an angle α with P, using the Triangle Law of Forces show that $R^2 = P^2 + Q^2 + 2PQ \cos\theta$ and that $\alpha = \tan^{-1}\left(\frac{Q\sin\theta}{P + Q\cos\theta}\right)$.

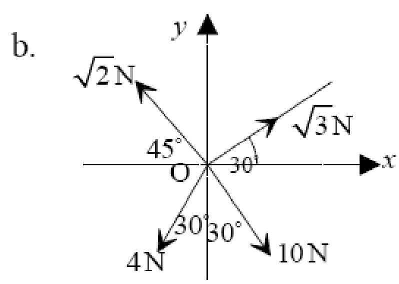
4. If the resultant of two forces P + Q and P – Q acting on a particle is $P^2 + Q^2$, show that the angle between the two forces is $\cos^{-1}\left(\frac{1}{2}.\frac{P^2+Q^2}{P^2-Q^2}\right)$.

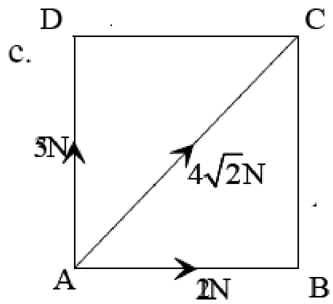
5. The magnitude of the resultant of two forces P and Q acting at a point at an angle θ, is P. When the direction of the force Q is reversed, the magnitude of the resultant is 2P. Show that $\frac{P^2}{Q^2} = \frac{2}{3}$ and determine the angle between P and Q.

6. Find
   i. the component in the Ox direction,
   ii. the component in the Oy direction,
   iii. the resultant

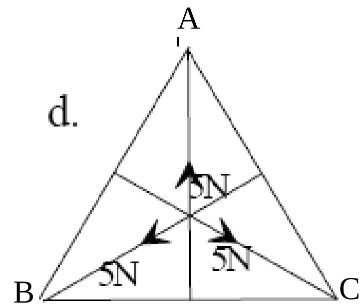
   of each of the following systems of forces.







ABCD is a square



ABC is an equilateral triangle

7. A system of co-planer forces 3N, 6N, $9\sqrt{3}$ N and 12N act on a particle. The angles between the first and second force, the second and third force and the third and fourth force are respectively 60^o, 90^o and 150^o. Determine the magnitude and the direction of the resultant.

8. The co-planer forces 3N, $8\sqrt{3}$ N, 2N, $3\sqrt{2}$ N and $2\sqrt{3}$ N act at a point. The first force is horizontal and the rest make angles of 60^o, 120^o, $225^o$and 300^o respectively with the horizontal. Find the resultant of the system.

9. One end of a light inextensible string is attached to a fixed point on a vertical wall while a particle of weight 4kg is attached to the other end. The system is kept in equilibrium in a vertical plane with the string making an angle of 30^o with the downward vertical, by applying a horizontal force S on the particle.

   (i) Determine the magnitude of S. (ii) Determine the tension in the string.

10. A particle P of mass 5kg is suspended from a horizontal ceiling by two light inextensible strings AP, PB. If the system is in equilibrium in a vertical plane with the string AP making an acute angle $\tan^{-1}\left(\frac{3}{4}\right)$ with the horizontal, and $A\hat{P}B = 90^o$, find the tensions in the strings.

11. If a particle of mass 1kg is kept at rest on a smooth fixed plane which makes an angle of $\tan^{-1}\left(\frac{3}{4}\right)$ with the horizontal, by applying a force parallel to the inclination of the plane, determine the magnitude of the force and the reaction between the plane and the particle. ($g = 10ms^{-2}$).

12. A light inextensible string is attached to two points A and B on a horizontal line, 24cm apart. If, when a particle of mass 56kg is attached to the mid-point C of the string, the system is in equilibrium with the point C being 7cm below the line AB, find the tensions in the string.

13. A light inextensible string ABCD is attached to two fixed points A and D which are in the same horizontal level. Two equal loads W are attached at the points B and C on the string. The system is in equilibrium with B being at a level higher than C and AB, CD making angles of 30^{o} and 60^{o} respectively with the downward vertical. Determine the tension in the various sections of the string and the inclination of BC to the vertical.

14. A light inextensible string ACDB is attached to two fixed points A and B which are in the same horizontal level, 7m apart. The lengths of AC, CD and DB are respectively $2\sqrt{6}$ m, 3m, 4m. A 1kg load is suspended from point C, and an X kg load is suspended from D. When the system is in equilibrium $C\hat{D}B$ is a right angle. Determine the value of X and the tensions in the string.

15. A light inextensible string ABCD is attached to two fixed points A and D which are in the same horizontal level. Two loads W_1 and W_2 are suspended from the points B and C respectively. The system is in equilibrium with B at a higher level than C and AB, BC, and CD making acute angles of θ, φ and β respectively with the upward vertical. Show that $\dfrac{W_1}{W_2} = \dfrac{\sin\beta\sin(\varphi-\theta)}{\sin\theta\sin(\varphi+\beta)}$.

7.0 Straight Lines

Competency	: 26	Uses the rectangular Cartesian coordinate system and related simple geometry results, suitable	
	27	Interprets the straight line in terms of Cartesian coordinates Grade 13 Teacher's Instructional Manual	
Competency Level	:	26.1, 26.2, 26.3, 27.1, 27.2, 27.3, 27.4, 27.5	
Subject Content	:	Straight Lines	

By studying this section you will develop the skills of

- describing the Cartesian coordinate system.
- defining the abscissa and ordinate.
- explaining how the signs of the abscissa and ordinate change in the four quadrants.
- finding the distance between two points on a Cartesian plane.
- determining the coordinates of the point that divides a line segment joining two given points in a given ratio, internally and externally.
- calculating the area of a triangle when the coordinates of the vertices are given.
- describing the locus of a point.
- defining the inclination of a straight line.
- defining the gradient of a straight line.
- expressing that two straight lines are parallel if and only if their gradients are equal.
- expressing that two straight lines which are not parallel to the coordinate axes are perpendicular if and only if the product of their gradients is equal to -1.
- finding the acute angle between two intersecting straight lines which are not perpendicular to each other.
- defining the intercept.
- representing the equation of a straight line in various forms.
- expressing the general form of a straight line.
- determining the coordinates of the point of intersection of two non-parallel lines.
- identifying the properties of parallel lines.
- writing down the equation of a straight line which is parallel to a given straight line.

- describing the properties of two lines which are perpendicular to each other.
- writing down the equation of a straight line which is perpendicular to a given line.
- expressing the equation of a straight line parametrically.
- determining the location of a point with respect to a given straight line.
- expressing the general form of a straight line passing through the point of intersection of two given straight lines.
- determining the tangent of the angle between two intersecting straight lines.
- determining the perpendicular distance from a given point to a given straight line.
- determining the area of a quadrilateral when the coordinates of the vertices are given.
- obtaining the equations of the bisectors of the angles between two intersecting lines.
- determining the coordinates of the image of a given point in a given straight line.
- determining the bisector of the angle between two straight lines that pass through the origin

Introduction

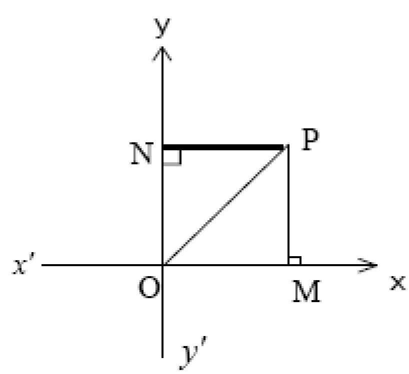
In coordinate geometry, results in pure geometry are proved using algebraic methods. Therefore, coordinate geometry is also called analytical geometry.

Accordingly, pure geometry and coordinate geometry are one. Therefore results from pure geometry can be used to solve problems in coordinate geometry.

The credit for introducing coordinate geometry is given to two French mathematicians Rene Descartes and Blaise Pascal. Descartes also introduced the Cartesian coordinate system.

7.1 The Cartesian Coordinate System

Two number lines $x'Ox$ and $y'Oy$ which are not parallel to each other are used to indicate the position of points in a plane. These two number lines are called the x-axis and the y-axis respectively. The point of intersection of the two axes, namely O, is called the origin, and the plane containing the two axes is called the Oxy plane (xy plane).
For convenience, the two lines $x'Ox$ and $y'Oy$ are taken to be perpendicular to each other. In this case, these axes are called rectangular Cartesian coordinate axes.

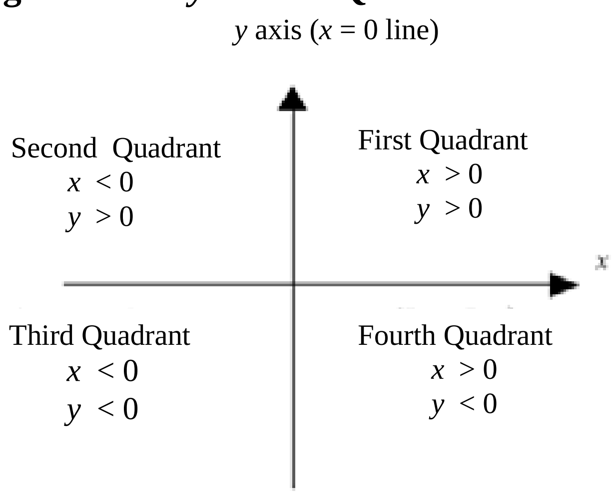


Let P be any point in the Oxy plane. Let M be the foot of the perpendicular dropped from P to the x'Ox axis and N be the foot of the perpendicular dropped from P to the y'Oy axis. If OM = x and ON = y, then the point P is represented by the ordered pair (x, y) called the Cartesian coordinates of P; and we write P ≡ (x, y). Any point P on the plane can be represented uniquely in the above manner by an ordered pair of real numbers. Conversely, corresponding to any ordered pair (x, y) of real numbers, there is a unique point P in the Oxy plane with coordinates (x, y).

7.2 The Abscissa and the Ordinate

If P ≡ (x, y), then x (= PN) is called the abscissa of P and y (= PM) is called the ordinate of P.

7.3 The Sign of *x* and *y* in each Quadrant



Since the y coordinate of any point on the x-axis is zero, the x-axis is also called the line $y = 0$, and since the x coordinate of any point on the y -axis is zero, the y-axis is also called the line $x = 0$. The region in the Oxy plane which does not belong to the two axes is divided into four sections by the two axes x'Ox and y'Oy. The region which contains the set of points P ≡ (x, y) with $x > 0$ and $y > 0$ is called the first quadrant, the region which contains the set of points P ≡ (x, y) with $x < 0$ and $y > 0$ is called the second quadrant, the region which contains the set of points P ≡ (x, y) with $x < 0$ and $y < 0$ is called the third quadrant and the region which contains the set of points P ≡ (x, y) with $x > 0$ and $y < 0$ is called the fourth quadrant of the Cartesian plane.

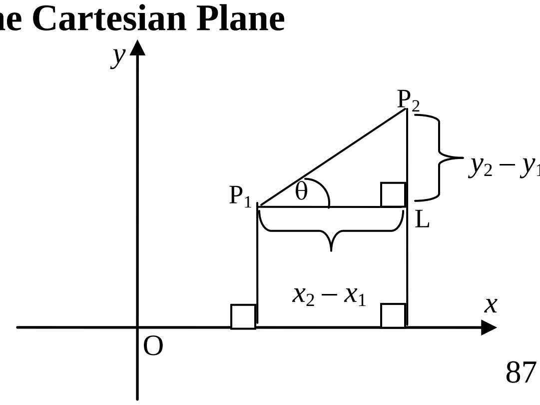
7.4 The Distance between Two Points in the Cartesian Plane

If $P_1 \equiv (x_1, y_1)$, $P_2 \equiv (x_2, y_2)$, then

$$(P_1P_2)^2 = (P_1L)^2 + (LP_2)^2$$

$$(P_1P_2)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



When $P_1 \equiv O$ and $P_2 \equiv P$; i.e., when $x_1 = 0, y_1 = 0, x_2 = x$ and $y_2 = y$, then the distance from the origin to the point $P \equiv (x, y)$ is $OP = \sqrt{x^2 + y^2}$.

Example

If $A \equiv (2,3),\ B \equiv (-2,1),\ C \equiv (0,-3),\ D \equiv (4,-1)$, show that ABCD is a square.

Solution

$$AB = \sqrt{(2+2)^2 + (3-1)^2} = \sqrt{16+4} = \sqrt{20}$$

$$BC = \sqrt{(-2-0)^2 + (1+3)^2} = \sqrt{4+16} = \sqrt{20}$$

$$CD = \sqrt{(0-4)^2 + (-3+1)^2} = \sqrt{16+4} = \sqrt{20}$$

$$DA = \sqrt{(4-2)^2 + (-1-3)^2} = \sqrt{4+16} = \sqrt{20}$$

$\therefore$ In the quadrilateral ABCD, AB = BC = CD = DA. $\therefore$ ABCD is a rhombus.

$$AC^2 = (2-0)^2 + (3+3)^2 = 4+36 = 40$$

$$AB^2 + BC^2 = 20 + 20 = 40$$

$$\therefore\ AC^2 = AB^2 + BC^2$$

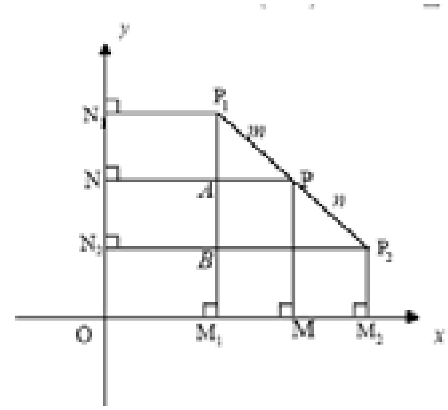
$\therefore A\hat{B}C = 90^o$.

Therefore, ABCD is a rhombus with $A\hat{B}C = 90^o$. Hence ABCD is a square.

7.5 The Coordinates of the Point which Divides the Line Segment joining the Points $P_1 \equiv (x_1, y_1)$ and $P_2 \equiv (x_2, y_2)$ in the Ratio $m : n$

Let $P_1 \equiv (x_1, y_1)$ and $P_2 \equiv (x_2, y_2)$, and suppose that the point P such that $P_1P : PP_2 = m : n$ is given by $P \equiv (\bar{x}, \bar{y})$.

ΔP_1AP and ΔP_1BP_2 are similar.



$$\therefore \frac{P_1A}{P_1B} = \frac{AP}{BP_2} = \frac{P_1P}{P_1P_2}$$

$$\therefore \frac{N_1N}{N_1N_2} = \frac{M_1M}{M_1M_2} = \frac{P_1P}{P_1P + PP_2}$$

$$\therefore \frac{ON_1 - ON}{ON_1 - ON_2} = \frac{OM - OM_1}{OM_2 - OM_1} = \frac{m}{m+n}$$

$$\Rightarrow \frac{y_1 - \bar{y}}{y_1 - y_2} = \frac{\bar{x} - x_1}{x_2 - x_1} = \frac{m}{m+n} \qquad \Rightarrow (m+n)y_1 - (m+n)\bar{y} = m(y_1 - y_2)$$

$$\Rightarrow \cancel{my_1} + ny_1 - \cancel{my_1} + my_2 = (m+n)\bar{y} \quad \Rightarrow \bar{y} = \frac{ny_1 + my_2}{(m+n)}$$

Similarly $\quad (m+n)\bar{x} - x_1(m+n) = m(x_2 - x_1)$

$$\Rightarrow mx_2 - \cancel{mx_1} + \cancel{mx_1} + nx_1 = (m+n)\bar{x} \quad \Rightarrow \bar{x} = \frac{nx_1 + mx_2}{(m+n)}$$

$$\therefore \; \mathrm{P} \equiv (\bar{x}, \bar{y}) \equiv \left[\frac{nx_1 + mx_2}{(m+n)}, \frac{ny_1 + my_2}{(m+n)}\right]$$

If $m = n = 1$, then the midpoint of the line segment joining $\mathrm{P}_1 \equiv (x_1, y_1)$ and $\mathrm{P}_2 \equiv (x_2, y_2)$ is obtained as $\mathrm{P} \equiv (\bar{x}, \bar{y}) = \left[\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right]$.

When the line segment is divided externally in the ratio m: n, one of m, n will be negative and the other will be positive.

Example

If A ≡ (12, -18) and B ≡ (-15, 24), determine the coordinates of the point which divides AB in the ratio 2:1

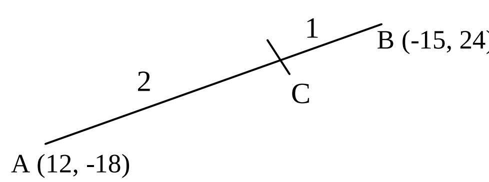
(i) internally. (ii) externally.

Solution

(i) Let C be the point which divides the line segment AB internally in the ratio 2:1.

$$\mathrm{C} \equiv \left[\frac{(1)(12)+(2)(-15)}{2+1}, \frac{(1)(-18)+(2)(24)}{2+1}\right]$$

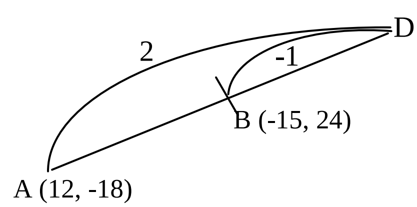
$$\equiv \left[\frac{12-30}{3}, \frac{-18+48}{3}\right] \equiv [-6, 10]$$



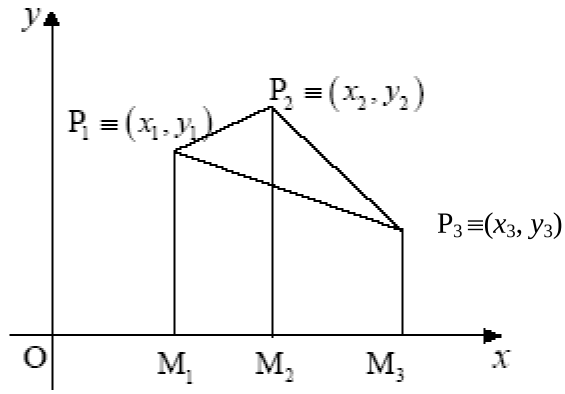
(ii) Let D be the point which divides the line segment AB externally in the ratio 2:1.

$$\mathrm{D} \equiv \left[\frac{(-1)(12)+(2)(-15)}{2+(-1)}, \frac{(-1)(-18)+(2)(24)}{2+(-1)}\right]$$

$$\equiv \left[\frac{-12-30}{1}, \frac{18+48}{1}\right] \equiv (-42, 66)$$



7.6 The Area of a Triangle



Let $P_1 \equiv (x_1, y_1)$, $P_2 \equiv (x_2, y_2)$ and $P_3 \equiv (x_3, y_3)$

Area of $\Delta P_1P_2P_3$ = (Area of trapezium $P_1M_1M_2P_2$
+ Area of trapezium $P_2M_2M_3P_3$
- Area of trapezium $P_1M_1M_3P_3$)

$\therefore$ Area of $\Delta P_1P_2P_3$

$$= \left| \left(\frac{y_1 + y_2}{2}\right)(x_2 - x_1) + \left(\frac{y_2 + y_3}{2}\right)(x_3 - x_2) - \left(\frac{y_1 + y_3}{2}\right)(x_3 - x_1) \right|$$

The modulus is taken for the area to be positive.

$\therefore$ Area of $\Delta P_1P_2P_3$

$$= \frac{1}{2}\left| (y_1 + y_2)(x_2 - x_1) + (y_2 + y_3)(x_3 - x_2) - (y_1 + y_3)(x_3 - x_1) \right|$$

$$= \frac{1}{2}\left| y_1x_2 - \cancel{y_1x_1} + \cancel{y_2x_2} - y_2x_1 + y_2x_3 - \cancel{y_2x_2} + \cancel{y_3x_3} - y_3x_2 - y_1x_3 + \cancel{y_1x_1} - \cancel{y_3x_3} + y_3x_1 \right|$$

$$= \frac{1}{2}\left| -(x_2y_3 - x_3y_2) + (x_1y_3 - x_3y_1) - (x_1y_2 - x_2y_1) \right| = \frac{1}{2}\left| (x_2y_3 - x_3y_2) - (x_1y_3 - x_3y_1) + (x_1y_2 - x_2y_1) \right|$$

It is convenient to remember this in the following manner.

Determinant Method

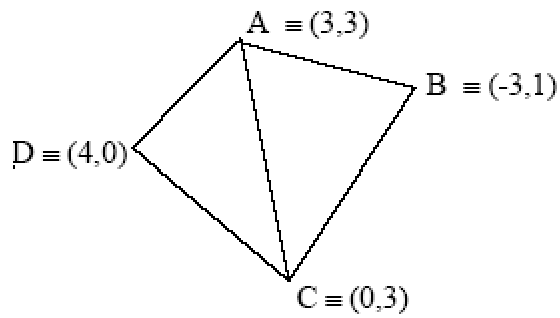
$$\therefore \text{ Area of } \Delta P_1P_2P_3 = \frac{1}{2}\left| \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \right| = \frac{1}{2}\left| \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} - \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \right|$$

$$= \frac{1}{2}\left| (x_2y_3 - x_3y_2) - (x_1y_3 - x_3y_1) + (x_1y_2 - x_2y_1) \right|$$

Example

Determine the area of the quadrilateral ABCD where $A \equiv (3,3)$, $B \equiv (-3,1)$, $C \equiv (0,3)$, $D \equiv (4,0)$
Determine also the ratio in which the diagonal AC divides the area of the quadrilateral.

Solution



The area of the triangle ABC

$$=\frac{1}{2}\left\| \begin{vmatrix} -3 & 0 \\ 1 & 3 \end{vmatrix} - \begin{vmatrix} 3 & 0 \\ 3 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -3 \\ 3 & 1 \end{vmatrix} \right\|$$

$= \frac{1}{2}$ |-9 – 9 + 12| = 3 square units

The area of the triangle ACD

$$=\frac{1}{2}\left\| \begin{vmatrix} 0 & 4 \\ 3 & 0 \end{vmatrix} - \begin{vmatrix} 3 & 4 \\ 3 & 0 \end{vmatrix} + \begin{vmatrix} 3 & 0 \\ 3 & 3 \end{vmatrix} \right\| = \frac{1}{2} |-12 + 12 + 9| = 4.5 \text{ square units}$$

$\therefore$ The area of the quadrilateral is 7.5 square units and the diagonal AC divides the area of the quadrilateral in the ratio 3: 4.5 = 6: 9 = 2:3

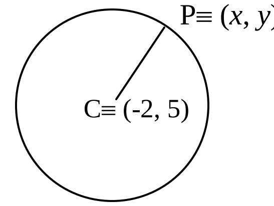
7.7 The Locus of a Point

If a point moves in a plane such that it satisfies a given geometric condition or several geometric conditions, then the path traced out by the point is called its locus.
If the point moves in the plane according to an equation, then that equation is called the equation of the locus of the point.

Example

Find the equation of a circle of radius 3 and centre C ≡ (-2, 5).

Solution



Let us consider an arbitrary point P ≡ (x, y) on the circle.
Then CP = 3 $\Rightarrow$ $CP^2 = 9$.
$(x+2)^2 + (y-5)^2 = 9 \;\; \Rightarrow \;\; x^2 + y^2 + 4x - 10y + 20 = 0$
Therefore, the equation of the circle of radius 3 and centre
C ≡ (-2, 5) is $x^2 + y^2 + 4x - 10y + 20 = 0$.

Exercise 1

1. For each of the following pairs of points, find the length of the line segment joining the two points.
   i. (2, 1) and (5, -1)   ii. ($b + c$, $c + a$) and ($c + a$, $a + b$)
   iii. ($\cos\alpha$, $-\sin\alpha$) and ($-\cos\alpha$, $\sin\alpha$)

2. If $A \equiv (6, 2)$, $B \equiv (-5, 0)$ and $C \equiv (-4, -3)$, show that ΔABC is an isosceles triangle.

3. Show that the points $A \equiv (7, 3)$, $B \equiv (3, 0)$, $C \equiv (0, -4)$ and $D \equiv (4, -1)$ are the vertices of a rhombus.

4. Determine the coordinates of the points of intersection of the line joining (1, -1) and (-1, 1), and the two axes.

5. Determine the coordinates of the point which divides the line segment joining the two points (5, 3) and (-2, 7) internally in the ratio 5:3.

6. For each of the following cases determine the coordinates of the points C and D which divide the line segment AB internally and externally in the ratio 2:1.
   i. $A \equiv (1,1)$, $B \equiv (4,-1)$ ii. $A \equiv (5,2)$, $B \equiv (-1,1)$ iii. $A \equiv (1,4)$, $B \equiv (3,6)$
   iv. $A \equiv (0,7)$, $B \equiv (3,5)$ v. $A \equiv (a,b)$, $B \equiv (a+c, b-c)$

7. Determine the lengths of the medians of the triangle with vertices (1, 1), (0, 2), (-1, -1).

8. In each of the following cases, if the given points are collinear, determine the value of k.
   i. (2, 3), (k, 6) and (3, 2).
   ii. (k, -1), (2, 4) and (5, 5)
   iii. (3, 2), (k, 3) and (1, 0)

9. i. If (a, 0), (0, b) and (1, 1) are collinear, show that $\frac{1}{a} + \frac{1}{b} = 1$.
   ii. If a, b and c are distinct, show that (a, a^2), (b, b^2) and (c, c^2) are non-collinear.

10. (6, 15) and (9, 1) are two adjacent vertices of a parallelogram. Determine the other two vertices such that the coordinates of the intersection point of the two diagonals is (1, 5).

11. For each of the following cases, determine the coordinates of the centroid of the triangle having the given points as vertices.
    i. (1, 2), (0, -2) and (2, -1)   ii. (3, 4), (5, 3) and (1, 5)
    iii. (5, -1), (-1, 4) and (-4, 3)

12. Determine the equation of the circle with radius 5 units and centre (3, -4).

13. A point moves such that the sum of the squares of the distance from the point to the y-axis and the distance from the point to (2, 1), is a constant. Determine the equation of the locus of the point.

14. A point moves such that the sum of the distances from the point to the points (3, 4) and (4, 3) is a constant. Determine the locus of the point.

15. i. Determine the equation of the set of points which are equi-distant from the two points (0, 2) and (3, 4).
    ii. Determine the equation of the set of points which are equi-distant from the point (1, 3) and the *x*-axis.

16. Determine the equation of the set of points which are equi-distant from $(a^2+b^2,\ a^2-b^2)$ and $(a^2-b^2,\ a^2+b^2)$.

17. i. The end points of a rod of length *p* lie on the coordinate axes. Determine the locus of the mid-point of the rod.
    ii. Determine the locus of the points which form a right angle with the line joining the points (1, 2) and (3, 4).

18. Show that the locus of the point that moves such that the sum of the distances from the point to the two points (*ae*, 0) and (-*ae*, 0) is equal to 2*a*, is given by $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Here $b^2=a^2(1-e^2)$.

19. Determine the equation of the circle with center (3, -4) and radius 5.

7.8 The inclination of a straight line

The angle that a straight line makes anti-clockwise from the positive direction of the *x*-axis is called the inclination of the straight line. The inclination θ of any straight line is such that $0 \le \theta < \pi$.

7.9 The gradient of a straight line

If any two points are selected on a straight line which is not parallel to the *y*-axis, then the ratio of the change in the *y* coordinate to the change in the *x* coordinate when moving from one point to the other, is called the gradient of the straight line.

The gradient of a straight line which is not parallel to the *y*-axis is a measure of the slope of the straight line with respect to the positive *x*-axis.

If $P_1 \equiv (x_1, y_1)$ and $P_2 \equiv (x_2, y_2)$ are any two points on a straight line which is not parallel to the y-axis,

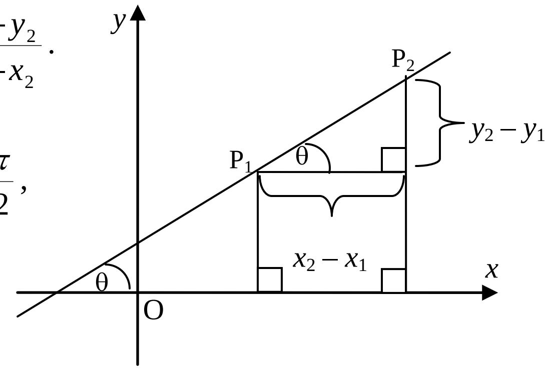
then the gradient of the straight line is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ or $\dfrac{y_1 - y_2}{x_1 - x_2}$.

Here $x_1 \neq x_2$.

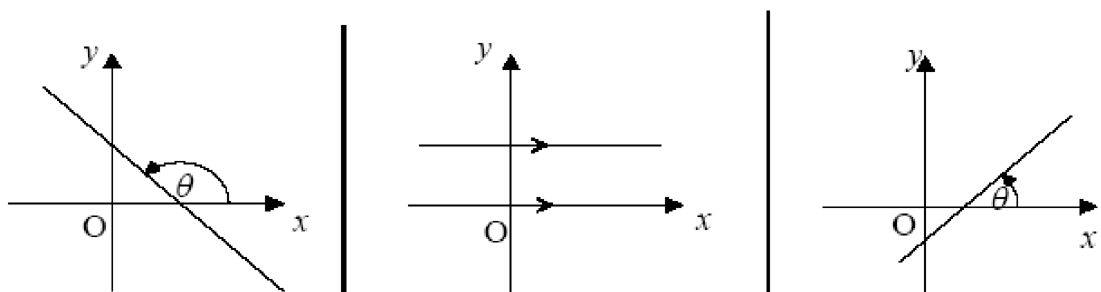
Also, if the inclination of the straight line is θ, when $\theta \neq \dfrac{\pi}{2}$,

$\tan\theta = \dfrac{y_2 - y_1}{x_2 - x_1}$.

Therefore $\tan\theta = m$ (gradient).



$\therefore$ The tangent of the angle that a straight line which is not parallel to the y-axis makes with the positive direction of the x-axis in an anti-clockwise direction can be defined as the gradient of the straight line.



$\dfrac{\pi}{2} < \theta < \pi$ | $\theta = 0$ | $0 < \theta < \dfrac{\pi}{2}$

gradient $m = \tan\theta < 0$ | gradient $m = \tan\theta = 0$ | gradient $m = \tan\theta > 0$

Two lines are parallel if and only if the gradients of the two lines are equal.

Proof

Let us take the inclination of the two lines as θ_1 and θ_2 and their gradients as $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$ respectively. $0 \leq \theta_1 < \pi$ and $0 \leq \theta_2 < \pi$.

Suppose the two lines are parallel. Then $\theta_1 = \theta_2$. $\therefore \tan\theta_1 = \tan\theta_2$, and hence $m_1 = m_2$. Therefore, the gradients of the two lines are equal.

If the gradients of the two lines are equal, then $m_1 = m_2$. $\therefore \tan\theta_1 = \tan\theta_2$. Since $0 \leq \theta_1, \theta_2 < \pi$, we obtain $\theta_1 = \theta_2$. Therefore, the two lines are parallel.

Therefore, two lines are parallel $\Leftrightarrow$ the gradients of the two lines are equal.

Two lines which are not parallel to the coordinate axes are perpendicular to each other if and only if the product of the gradients of the two lines equals -1.

Proof

Let us take the inclination of the two lines as θ_1 and θ_2 and their gradients as $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$ respectively. $0 < \theta_1 < \pi$ and $0 < \theta_2 < \pi$.

Suppose the two lines are perpendicular to each other.

Then,

$$\theta_1 - \theta_2 = \frac{\pi}{2} \quad \text{or} \quad \theta_2 - \theta_1 = \frac{\pi}{2} \Rightarrow \quad \theta_1 = \frac{\pi}{2} + \theta_2 \quad \text{or} \quad \theta_2 = \frac{\pi}{2} + \theta_1$$

$$\Rightarrow \quad \tan\theta_1 = \tan\left(\frac{\pi}{2} + \theta_2\right) \quad \text{or} \quad \tan\theta_2 = \tan\left(\frac{\pi}{2} + \theta_1\right)$$

$$\Rightarrow \quad \tan\theta_1 = -\cot\theta_2 \quad \text{or} \quad \tan\theta_2 = -\cot\theta_1$$

$$\Rightarrow \quad \tan\theta_1 = -\frac{1}{\tan\theta_2} \quad \text{or} \quad \tan\theta_2 = -\frac{1}{\tan\theta_1}$$

$$\Rightarrow \quad \tan\theta_1 \tan\theta_2 = -1 \qquad \Rightarrow m_1 . m_2 = -1$$

Therefore, the product of the gradients is -1, of two perpendicular lines which are not parallel to the axes.

Now suppose that the gradient of two perpendicular lines which are not parallel to the axes is -1.

Then $\quad m_1 m_2 = -1 \quad , \quad m_1, m_2 \neq 0$

$$\Rightarrow \quad m_1 = -\frac{1}{m_2} \quad , \quad m_1, m_2 \neq 0$$

$$\Rightarrow \quad \tan\theta_1 = -\frac{1}{\tan\theta_2} \quad , \quad \theta_1, \theta_2 \neq 0$$

$$\Rightarrow \quad \tan\theta_1 = -\cot\theta_2 \quad , \quad \theta_1, \theta_2 \neq 0$$

$$\Rightarrow \quad \tan\theta_1 = \tan\left(\frac{\pi}{2} + \theta_2\right) \quad , \quad \theta_1, \theta_2 \neq 0$$

$$\theta_1 = n\pi + \left(\frac{\pi}{2} + \theta_2\right) \quad \Rightarrow \theta_1 - \theta_2 = n\pi + \frac{\pi}{2} \quad , n \in Z \text{ and } 0 < \theta_1 \, , \ \theta_2 < \pi$$

$$\therefore \quad n = 0 \quad \Rightarrow \quad \theta_1 - \theta_2 = \frac{\pi}{2} \text{ ------}(1)$$

$$n = -1 \quad \Rightarrow \quad \theta_1 - \theta_2 = -\pi + \frac{\pi}{2} = -\frac{\pi}{2} \quad \Rightarrow \quad \theta_2 - \theta_1 = \frac{\pi}{2} \text{ ----------(2)}$$

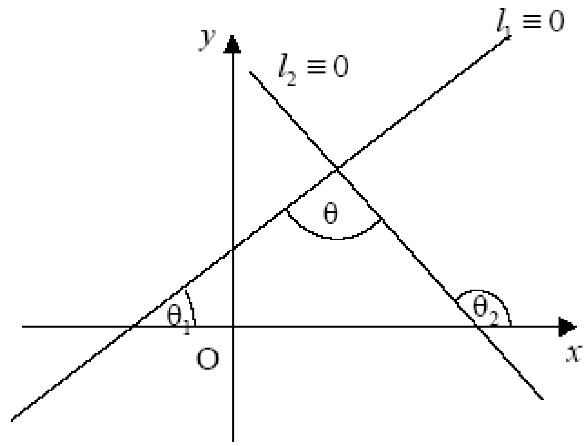
$\therefore$ From (1) and (2) $\quad \theta_1 = \frac{\pi}{2} + \theta_2 \quad$ or $\quad \theta_2 = \frac{\pi}{2} + \theta_1, \; \theta_1, \theta_2 \neq 0$

Therefore, the two lines are perpendicular to each other.

If α is the angle between two lines with gradients m_1 and m_2, which are not perpendicular to each other, then $\tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$**.**

Proof

Suppose the two straight lines are $l_1 = 0$ and $l_2 = 0$ and that the gradients of the two lines are $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$ respectively. From the figure,



$$\theta + \theta_1 = \theta_2 \quad \Rightarrow \quad \theta = \theta_2 - \theta_1$$

$$\tan\theta = \tan(\theta_2 - \theta_1)$$

$$= \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2 \tan\theta_1}$$

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$$

$$\tan(\pi - \theta) = -\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

The tangent of the angle between the two lines $l_1 = 0$ and $l_2 = 0$ is $\frac{m_1 - m_2}{1 + m_1 m_2}$ or $\frac{m_2 - m_1}{1 + m_1 m_2}$.

The tangent of an acute angle should be positive.
Therefore, if the acute angle between the two straight lines $l_1 = 0$ and $l_2 = 0$ is α, then $\tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$.

Example

Determine if they are defined, the gradients of the straight lines passing through the following pairs of points. Determine also the inclination of the lines.

i. A $(1,0)$, B$\left(2, \frac{1}{\sqrt{3}}\right)$ ii. E$(1,4)$, F $(1, -3)$

Solution

i. The gradient of AB, $m_1 = \dfrac{0-\frac{1}{\sqrt{3}}}{1-2} = \dfrac{\left(-\frac{1}{\sqrt{3}}\right)}{-1} = \dfrac{1}{\sqrt{3}}$ and therefore, the inclination of AB is $\theta_1 = \dfrac{\pi}{6}$.

ii. The x coordinates of both E and F are equal to 1. Therefore EF is parallel to the y-axis. Hence the gradient of EF is not defined. The inclination of EF is $\theta_2 = \dfrac{\pi}{2}$.

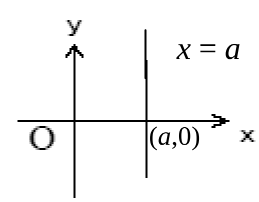
7.10 Intercept

The intercept of a straight line which is not parallel to the y-axis is the y coordinate of the point of intersection of the straight line and the y-axis.

7.11 Various forms of the equation of a straight line

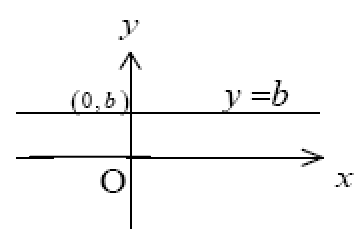
(i) The equation of the straight line passing through the point (a, 0), parallel to the y-axis

While y can take any real number, the x coordinate of every point on the straight line is a. Therefore, the equation of the straight line is $x = a$. On the y-axis, while y can take any real number, the x coordinate of every point is zero. Therefore, the equation of the y-axis is $x = 0$.



(ii) The equation of the straight line passing through the point (0, b), parallel to the x-axis

While x can take any real number, the y coordinate of every point on the straight line is b. Therefore, the equation of the straight line is $y = b$. On the x-axis, while x can take any real number, the y coordinate of every point is zero. Therefore, the equation of the x-axis is $y = 0$.

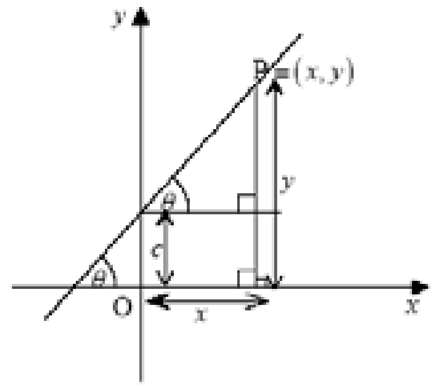


(iii) The equation of the straight line with gradient *m* and intercept *c*

Let P ≡ (x, y) be an arbitrary point on the straight line.

If the gradient is $m = \tan\theta$, then $m = \frac{y-c}{x}$.

Therefore, $mx = y - c$. Hence $y = mx + c$.
Conversely, the gradient and the intercept of a straight line represented by $y = mx + c$, is m and c respectively. The equation of the straight line with gradient m which passes through the origin is $y = mx$. $(c = 0)$

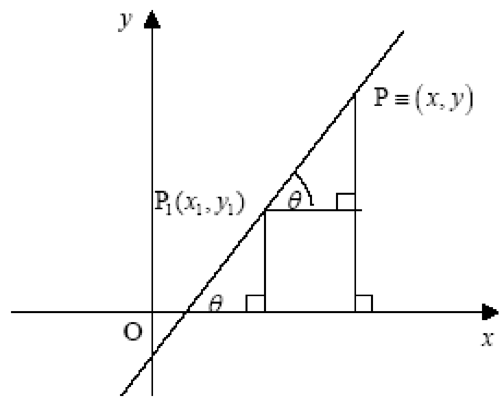


(iv) The equation of the straight line with gradient *m* passing through the point $P_1 \equiv (x_1, y_1)$

Let P ≡ (x, y) be an arbitrary point on the straight line.

If the gradient is $m = \tan\theta$, then $m = \frac{y-y_1}{x-x_1}$.

Therefore, $y - y_1 = m(x - x_1)$.
Thus the equation of the straight line with gradient m, passing through the point (x_1, y_1) is of the form $y - y_1 = m(x - x_1)$

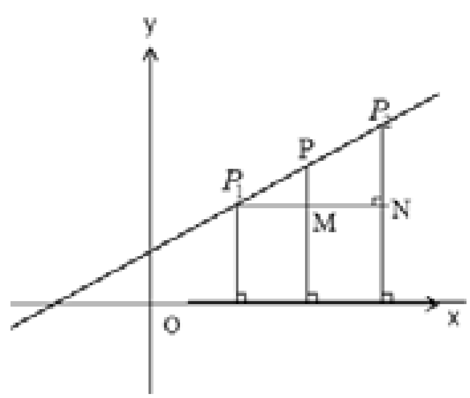


(v) The equation of the straight line passing through the two points $P_1 \equiv (x_1, y_1)$ and $P_2 \equiv (x_2, y_2)$

If P_1 and P_2 are distinct, then $x_1 \neq x_2$.
Let P ≡ (x, y) be an arbitrary point on the straight line.
The two triangles P_1PM and P_1P_2N are similar.

Therefore, $\frac{PM}{P_1M} = \frac{P_2N}{P_1N} \Rightarrow \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$

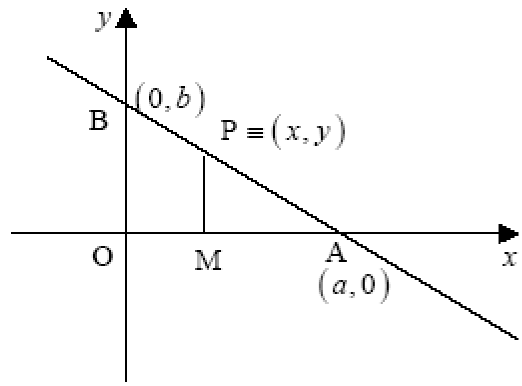
Hence $y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$



When $x_1 = x_2$, the equation of the straight line is $x = x_1$.

(vi) The equation of the straight line with *x*-intercept *a* and *y*-intercept *b*.

Let P ≡ (x, y) be an arbitrary point on the straight line.
ΔOAB and ΔMAP are similar.



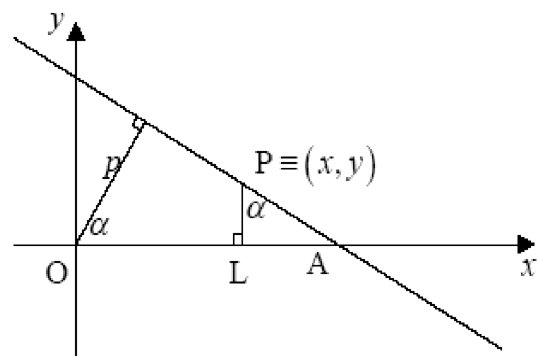
$$\frac{\text{OB}}{\text{OA}} = \frac{\text{PM}}{\text{MA}} \quad \Rightarrow \quad \frac{b}{a} = \frac{y}{a - x}$$

$$\therefore \; \frac{a - x}{a} = \frac{y}{b} \;\Rightarrow \therefore \; \frac{x}{a} + \frac{y}{b} = 1 \;.$$

Conversely the equation $\frac{x}{a}+\frac{y}{b}=1$ represents the straight line with x-intercept a and y-intercept b.

(vii) The equation of a straight line such that the perpendicular to it from the origin O has an inclination α and length *p*. (Normal Form)

Let P ≡ (x, y) be an arbitrary point on the straight line.



$$\cos\alpha = \frac{p}{\text{OA}} \qquad \therefore \; \text{OA} = \frac{p}{\cos\alpha}$$

From Δ APL,

$$\tan\alpha = \frac{\text{LA}}{\text{PL}} = \frac{\text{OA} - \text{OL}}{\text{PL}} \;.$$

$$\tan\alpha = \frac{\dfrac{p}{\cos\alpha} - x}{y} \;\Rightarrow\; y \;\tan\alpha = \frac{p}{\cos\alpha} - x$$

$$\Rightarrow \; y \;\frac{\sin\alpha}{\cos\alpha} = \frac{p}{\cos\alpha} - x \;\Rightarrow\; y \;\sin\alpha = p - x\cos\alpha$$

$$\Rightarrow x\cos\alpha + y\sin\alpha = p$$

Since p is the perpendicular distance from O it is always non-negative.

Example

Determine the equation of the straight line satisfying the given conditions.

i. Straight line through (2, 3), parallel to the x-axis.
ii. Straight line through (2, 3), parallel to the y-axis.
iii. Straight line with gradient -3 and intercept 4.
iv. Straight line through (2, 3) with gradient 5.
v. Straight line with x-intercept 3 and y-intercept 5.
vi. Straight line such that the perpendicular to it from the origin is of length $\sqrt{2}$ and has an inclination of 45^o.

Solution

i. Using the form $y = b$, the equation of the straight line through (2, 3), parallel to the x-axis is $y = 3$; i.e., $y - 3 = 0$.
ii. Using the form $x = a$, the equation of the straight line through (2, 3), parallel to the y-axis is $x = 2$; i.e., $x - 2 = 0$.
iii. Using the form $y = mx + c$, the equation of the straight line with gradient -3 and intercept 4 is $y = -3x + 4$; i.e., $3x + y - 4 = 0$.
iv. Using the form $y - y_1 = m(x - x_1)$, the equation of the straight line through (2, 3) with gradient 5 is $y - 3 = 5(x - 2)$; i.e., $5x - y - 7 = 0$.
v. Using the form $\frac{x}{a}+\frac{y}{b}=1$, the equation of the straight line with with x-intercept 3 and y-intercept 5 is $\frac{x}{3}+\frac{y}{5}=1$; i.e., $5x + 3y - 15 = 0$.
vi. Using the form $x\cos\alpha + y\sin\alpha = p$, the equation of the straight line such that the perpendicular to it from the origin is of length $\sqrt{2}$ and has an inclination of 45^o, is $x\cos 45^o + y\sin 45^o = \sqrt{2}$; i.e., $x + y - 2 = 0$.

The general form of the equation of a straight line is Ax + By + C = 0, where $A^2 + B^2 \neq 0$.

Proof

Let $P_1 \equiv (x_1, y_1)$ and $P_2 \equiv (x_2, y_2)$ be any two points on the curve represented by the equation Ax + By + C = 0.
Then since $P_1 \equiv (x_1, y_1)$ lies on the curve, $Ax_1 + By_1 + C = 0$ -------------- (1)
Similarly since $P_2 \equiv (x_2, y_2)$ lies on the curve, $Ax_2 + By_2 + C = 0$ -------------- (2)

$$(2) - (1) \Rightarrow \quad A(x_2 - x_1) + B(y_2 - y_1) = 0$$

$$B(y_2 - y_1) = -A(x_2 - x_1)$$

If $x_1 \neq x_2$, $$B\frac{(y_2 - y_1)}{(x_2 - x_1)} = -A$$

If B $\neq$ 0, $\dfrac{y_2 - y_1}{x_2 - x_1} = -\dfrac{\text{A}}{\text{B}}$ $\Rightarrow$ the gradient of P_1P_2 is a constant.

P_1, P_2 are any two points on the curve represented by the equation Ax + By + C = 0.
For all B $\neq$ 0, the gradient of P_1P_2 is a constant. Thus the equation Ax + By + C = 0 represents a straight line. The general equation of a straight line is Ax + By + C = 0.
Here, A, B, C are real constants with A and B not equal to zero together; i.e., $A^2 + B^2 \neq 0$.

If B = 0, then A $\neq$ 0.

Then Ax + By + C = 0 $\Rightarrow$ Ax + C = 0 $\Rightarrow$ $x = -\dfrac{\text{C}}{\text{A}}$.

In this case Ax + By + C = 0 represents the equation of a straight line which is parallel to the y-axis.

Example

If a straight line is given by Ax + By + C = 0, where both A and B are not zero together, find

i. the gradient and intercept
ii. the x and y intercepts
iii. the distance from the origin to the straight line and the inclination of the perpendicular to the straight line through the origin,

Solution

i. Ax + By + C = 0

By = -Ax – C

<u>If B $\neq$ 0</u>, $y = -\dfrac{\text{A}}{\text{B}}x - \dfrac{\text{C}}{\text{B}}$

$\therefore$ the gradient = $-\dfrac{\text{A}}{\text{B}}$

The intercept = $-\dfrac{\text{C}}{\text{B}}$

<u>If B = 0</u>, then A $\neq$ 0.

Ax + C = 0 $\Rightarrow$ $x = -\dfrac{\text{C}}{\text{A}}$. In this case the straight line is parallel to the y – axis, and the gradient and the intercept are not defined.

ii. Ax + By + C = 0

Ax + By = - C

(a) When A ≠ 0, B ≠ 0, C ≠ 0.

$$\frac{\mathrm{A}}{\text{-C}}x+\frac{\mathrm{B}}{\text{-C}}y=1 \Rightarrow \frac{x}{(\text{-C/A})}+\frac{y}{(-\mathrm{C/B})}=1.$$

Therefore, the x-intercept is –C/A and the y-intercept is –C/B.

(b) When C = 0, A ≠ 0, B ≠ 0.

Then the equation of the straight line is Ax + By = 0, and it passes through the origin (0, 0). Therefore the x intercept is 0 and the y intercept is also 0.

(c) When A = 0, B ≠ 0, C ≠ 0.

Then the equation of the straight line is By + C = 0; i.e., $y = -\frac{\mathrm{C}}{\mathrm{B}}$. Therefore the straight line is parallel to the x axis. The x intercept is not defined, the y intercept is $-\frac{\mathrm{C}}{\mathrm{B}}$.

(d) When B = 0, C ≠ 0, A ≠ 0.

Then the equation of the straight line is Ax + C = 0; i.e., $x = -\frac{\mathrm{C}}{\mathrm{A}}$. Therefore the straight line is parallel to the y-axis. The y intercept is not defined, the x intercept is $-\frac{\mathrm{C}}{\mathrm{A}}$.

iii. Ax + By + C = 0

Since $\sqrt{\mathrm{A}^2+\mathrm{B}^2} \neq 0$, $\frac{\mathrm{A}}{\sqrt{\mathrm{A}^2+\mathrm{B}^2}}x+\frac{\mathrm{B}}{\sqrt{\mathrm{A}^2+\mathrm{B}^2}}y+\frac{\mathrm{C}}{\sqrt{\mathrm{A}^2+\mathrm{B}^2}}=0$

$$x\left(\frac{\mathrm{A}}{\sqrt{\mathrm{A}^2+\mathrm{B}^2}}\right)+y\left(\frac{\mathrm{B}}{\sqrt{\mathrm{A}^2+\mathrm{B}^2}}\right)=\frac{-\mathrm{C}}{\sqrt{\mathrm{A}^2+\mathrm{B}^2}}$$

When $\mathrm{C}>0$, $x\left(\frac{-\mathrm{A}}{\sqrt{\mathrm{A}^2+\mathrm{B}^2}}\right)+y\left(\frac{-\mathrm{B}}{\sqrt{\mathrm{A}^2+\mathrm{B}^2}}\right)=\frac{\mathrm{C}}{\sqrt{\mathrm{A}^2+\mathrm{B}^2}}>0$

There exists α such that $\left(\frac{-\mathrm{A}}{\sqrt{\mathrm{A}^2+\mathrm{B}^2}}\right)=\cos\alpha$ $\left(\frac{-\mathrm{B}}{\sqrt{\mathrm{A}^2+\mathrm{B}^2}}\right)=\sin\alpha$.

$x\cos\alpha+y\sin\alpha=p$, here $p=\frac{\mathrm{C}}{\sqrt{\mathrm{A}^2+\mathrm{B}^2}}$

The length of the perpendicular from the origin to the straight line is $p = \frac{C}{\sqrt{A^2+B^2}}$.

The inclination of the perpendicular is α such that

$$\left(\frac{-A}{\sqrt{A^2+B^2}}\right) = \cos\alpha \qquad \left(\frac{-B}{\sqrt{A^2+B^2}}\right) = \sin\alpha$$

When C < 0,

$$x\left(\frac{A}{\sqrt{A^2+B^2}}\right) + y\left(\frac{B}{\sqrt{A^2+B^2}}\right) = \frac{-C}{\sqrt{A^2+B^2}} > 0$$

There exists α such that

$$\left(\frac{A}{\sqrt{A^2+B^2}}\right) = \cos\alpha \qquad \left(\frac{B}{\sqrt{A^2+B^2}}\right) = \sin\alpha$$

$x\cos\alpha + y\sin\alpha = p$, where $p = \frac{-C}{\sqrt{A^2+B^2}}$.

The length of the perpendicular from the origin to the straight line is $p = \frac{-C}{\sqrt{A^2+B^2}}$.

The inclination of the perpendicular is α such that

$$\left(\frac{A}{\sqrt{A^2+B^2}}\right) = \cos\alpha \qquad \left(\frac{B}{\sqrt{A^2+B^2}}\right) = \sin\alpha$$

7.12 The Parametric Form of a Straight Line

If the x and y coordinates of an arbitrary point on a line are expressed in terms of the same parameter, we call it the parametric form.

Several special cases are indicated below.

(i) The parametric form when A ≡ (x_0, y_0) is a point on the straight line given by $ax + by + c = 0$**.** $(a \neq 0, b \neq 0)$

Let P ≡ (x, y) be an arbitrary point on the straight line.
Since P ≡ (x, y) is on the given line, $ax + by + c = 0$ -------------- (1)
Since A ≡ (x_0, y_0) is on the given line, $ax_0 + by_0 + c = 0$ ---------------(2)

(1), (2) $\Rightarrow \dfrac{x-x_0}{-b} = \dfrac{y-y_0}{a} = t$, where t is a parameter.

Therefore, when A $\equiv$ (x_0, y_0) is a point on the straight line given by $ax+by+c=0$, the parametric form of the equation is $x = x_0 - bt, \quad y = y_0 + at$.

$$\text{AP} = \sqrt{(x-x_0)^2 + (y-y_0)^2} = \sqrt{b^2t^2 + a^2t^2} = \sqrt{b^2+a^2} \quad |t|$$

If a, b are chosen such that $a^2+b^2=1$, then AP = $|t|$

Example

Determine the two points which are 2 units from the point A $\equiv$ (1, -1) on the straight line given by $3x-4y-7=0$.

Solution

Let P $\equiv$ (x, y) be an arbitrary point on the straight line, $3x-4y-7=0$ ------------ (1)
Since A $\equiv$ (1, -1) is a point on the straight line, 3 + 4 – 7 = 0 ------------ (2)
From (1) – (2) we obtain $3(x-1)-4(y+1)=0$

$$\Rightarrow \quad 3(x-1) = 4(y+1) \quad \Rightarrow \quad \frac{x-1}{4} = \frac{y+1}{3}$$

$$\therefore \quad \frac{x-1}{\left(\frac{4}{5}\right)} = \frac{y+1}{\left(\frac{3}{5}\right)} = t \quad \text{(a}$$

$$\therefore \quad (x-1) = \frac{4}{5}t \;; \qquad (y+1) = \frac{3}{5}t$$

$$\therefore \quad x = 1+\frac{4}{5}t \;; \qquad y = -1+\frac{3}{5}t$$

If AP = $|\,t\,|$, for points P on the straight line which are 2 units from A, $|\,t\,|$ = 2.
$\therefore t = \pm 2$.
When t = 2,

$$x = 1+\frac{4}{5}\times 2 = 1+\frac{8}{5} = \frac{13}{5} \quad , \quad y = -1+\frac{3}{5}\times 2 = -1+\frac{6}{5} = \frac{1}{5}$$

In this case the point which is 2 units from A is $\left(\dfrac{13}{5}, \dfrac{1}{5}\right)$.

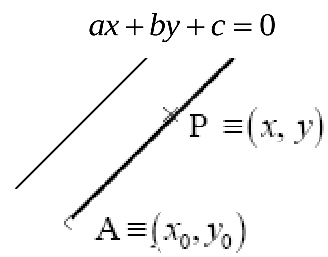
When t = -2,

$$x = 1+\frac{4}{5}\times(-2) = 1-\frac{8}{5} = -\frac{3}{5} \quad , \quad y = -1+\frac{3}{5}\times(-2) = -1-\frac{6}{5} = -\frac{11}{5}$$

In this case the point which is 2 units from A is $\left(-\frac{3}{5},-\frac{11}{5}\right)$

Therefore, the two points which are 2 units from the point A ≡ (1, -1) on the straight line given by $3x-4y-7=0$ are $\left(\frac{13}{5},\frac{1}{5}\right)$ and $\left(-\frac{3}{5},-\frac{11}{5}\right)$.

(ii) The parametric form of the straight line which passes through A ≡ (x_0, y_0) and is parallel to the straight line given by $ax+by+c=0$. ($a \neq 0$, $b \neq 0$)



Let P ≡ (x, y) be an arbitrary point on the straight line.

The gradient of the line AP

= the gradient of the line $ax+by+c=0$

$$\frac{y-y_0}{x-x_0} = -\frac{a}{b} \Rightarrow \frac{x-x_0}{b} = \frac{y-y_0}{-a} = t \quad \text{(a parameter)}$$

$$\Rightarrow \frac{x-x_0}{b} = t, \quad \frac{y-y_0}{-a} = t \Rightarrow x-x_0 = bt, \quad y-y_0 = -at$$

$$\Rightarrow x = x_0 + bt, \quad y = y_0 - at$$

Therefore, the parametric form of the equation is $x = x_0 + bt, \quad y = y_0 - at$.

Here too, if a, b are chosen such that $a^2+b^2=1$, then AP = | t |.

Then $\frac{x-x_0}{-b} = \frac{y-y_0}{a}$ can be written as $\frac{x-x_0}{\frac{-b}{\sqrt{a^2+b^2}}} = \frac{y-y_0}{\frac{a}{\sqrt{a^2+b^2}}} = t$

$$\therefore \frac{x-x_0}{\frac{-b}{\sqrt{a^2+b^2}}} = t \Rightarrow x-x_0 = \frac{-b}{\sqrt{a^2+b^2}}\, t \Rightarrow x = x_0 + \frac{-b}{\sqrt{a^2+b^2}}\, t$$

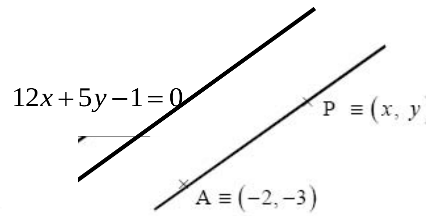
and $\frac{y-y_0}{\frac{a}{\sqrt{a^2+b^2}}} = t \Rightarrow y-y_0 = \frac{a}{\sqrt{a^2+b^2}}\, t \Rightarrow y = y_0 + \frac{a}{\sqrt{a^2+b^2}}\, t$

Example

Determine the two points which are on the line through A ≡ (-2, -3) parallel to $12x+5y-1=0$, and 26 units from A.

Solution

Let P ≡ (x, y) be an arbitrary point on the straight line.
The gradient of the line AP
= the gradient of the line $12x+5y-1=0$



$$\therefore \quad \left(\frac{y+3}{x+2}\right) = \frac{-12}{5} \quad \Rightarrow \quad \frac{x+2}{5} = \frac{y+3}{-12}$$

$$\Rightarrow \frac{x+2}{5/13} = \frac{y+3}{-12/13} = t \quad \text{(a parameter)}$$

$$\Rightarrow x+2 = \frac{5}{13}t \quad \& \quad y+3 = -\frac{12}{13}t \;\Rightarrow x = -2+\frac{5}{13}t \quad \& \quad y = -3-\frac{12}{13}t$$

For points which are 26 units from A ≡ (2, 3), | t | = 26.

$$|t| = 26 \quad \Rightarrow \quad t = \pm 26 \quad \Rightarrow \quad t = 26 \quad \text{or} \quad t = -26$$

$$t = 26 \;\Rightarrow x = -2+\frac{5}{13}\times 26 \quad \text{and} \quad y = -3-\frac{12}{13}\times 26$$

$$\Rightarrow x = -2+5\times 2 \;\text{ and }\; y = -3-12\times 2 \;\Rightarrow x=8 \;\;\&\;\; y=-27$$

$$\therefore\; t = 26 \;\Rightarrow\; \text{P} \equiv (8, -27)$$

$$t = -26 \;\Rightarrow\; x = -2+\frac{5}{13}\times(-26) \;\text{ and }\; y = -3-\frac{12}{13}\times(-26)$$

$$\Rightarrow\; x = -2-5\times 2 \;\text{ and }\; y = -3+12\times 2$$

$$\Rightarrow\; x = -12 \;\text{ and }\; y = 21$$

$\therefore$ when t = -26, P ≡ (-12, 21).

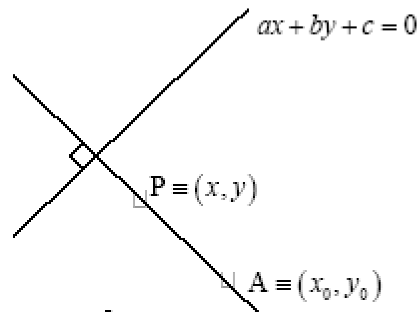
The two points on the line through A parallel to $12x+5y-1=0$ and 26 units from A are (8, -27) and (-12, 21).

(iii) The parametric form of the straight line through A ≡ (x_0, y_0) perpendicular to the straight line given by $ax+by+c=0$. $(a \neq 0,\ b \neq 0)$

Let P ≡ (x, y) be an arbitrary point on the required straight line.
Then, (gradient of AP) × (gradient of $ax+by+c=0$) = -1.

$$\left(\frac{y-y_0}{x-x_0}\right)\times\left(\frac{-a}{b}\right)=-1 \Rightarrow \left(\frac{y-y_0}{x-x_0}\right)\times\left(\frac{a}{b}\right)=1$$

Let $\dfrac{y-y_0}{b}=\dfrac{x-x_0}{a} = t$ (a parameter)



Since, $\dfrac{y-y_0}{b}=t$, $\dfrac{x-x_0}{a} = t$

$x-x_0=at$ and $y-y_0=bt$

Therefore, $x=x_0+at$ and $y=y_0+bt$

Therefore, the parametric form of the equation is $x=x_0+at$ and $y=y_0+bt$ where t is a parameter.

Example

Determine the two points on the straight line through A ≡ (-2, 5) perpendicular to the straight line $6x-8y+3=0$ and 10 units from A.

Solution

Let P ≡ (x, y) be an arbitrary point on the required straight line.
Then, (gradient of AP) × (gradient of $6x-8y+3=0$) = -1.

$$\therefore \left(\frac{y-5}{x+2}\right)\times\left(\frac{6}{8}\right) = -1 \Rightarrow \frac{x+2}{3} = \frac{y-5}{-4}$$

$$\Rightarrow \frac{x+2}{3/5} = \frac{y-5}{-4/5} = t \text{ (a parameter)}$$

$$\Rightarrow x+2 = \frac{3}{5}t \ \ \& \ \ y-5 = -\frac{4}{5}t \Rightarrow x = -2+\frac{3}{5}t \ \ \& \ \ y = 5-\frac{4}{5}t$$

For points which are 10 units from A ≡ (-2, 5), | t | = 10.

$|\,t\,| = 10 \quad \Rightarrow \quad t = \pm 10 \quad \Rightarrow \quad t = 10$ or $t = -10$

$t = 10 \quad \Rightarrow \quad x = -2 + \frac{3}{5} \times 10$ and $y = 5 - \frac{4}{5} \times 10$

$\Rightarrow x = -2 + 3 \times 2$ & $y = 5 - 4 \times 2 \Rightarrow x = 4$ & $y = -3$

$\therefore \; t = 10 \quad \Rightarrow \quad P \equiv (4, -3)$

$t = -10 \quad \Rightarrow \quad x = -2 + \frac{3}{5} \times (-10)$ & $y = 5 - \frac{4}{5} \times (-10)$

$\Rightarrow x = -2 - 3 \times 2$ and $y = 5 + 4 \times 2$

$\Rightarrow x = -8$ and $y = 13$

$\therefore \; t = -10 \quad \Rightarrow \quad P \equiv (-8, 13)$

Therefore, the two points which are on the line through A ≡ (-2, 5), perpendicular to the straight line $6x - 8y + 3 = 0$ and 10 units from A are (4, -3) and (-8, 13).

Exercise 2

1. (a) For each of the following cases, determine the gradient of the straight line passing through the given pair of points, if it is defined. Determine also the inclination of the line joining the two points.

   i. A(2, 1), B(3, 2)   ii. E(1, 4), F(1, -3)

   iii. G(-1, 0), H(0, - $\sqrt{3}$)   iv. I(3, $\sqrt{3}$), J(-9, $\sqrt{3}$)

   (b) Find the acute angle between

   i. AB and EF   ii. EF and GH   iii. GH and IJ

2. For each of the following cases, determine the equation of the straight line satisfying the given conditions.

   i. (a) -3 units away from the *x*-axis   (b) 6 units from the *y*-axis
   ii. gradient of -2 and passing through the origin
   iii. having an inclination of 60^{o} and passing through the origin
   iv. (a) gradient of 5 and intercept 4   (b) gradient 0 and intercept -7
   v. Through the points (a) $P_1(2, 3)$ and $P_2(3, -1)$ (b) $P_1(1, -10)$ and $P_2(1, 2)$
   vi. With *x*- intercept $a + b$ and *y*-intercept $a - b$

3. Determine the *x*-intercept and the *y*-intercept of the straight lines given by each of the following equations.

   i. $4x + 3y - 25 = 0$ ii. $6x - 8y + 24 = 0$

   iii. $x - 5y - 10 = 0$ iv. $8x - 10y + 6 = 0$

4. For each of the following cases, determine the equation of the line, its gradient and its intercept, if the length of the perpendicular from the origin to the given line is p, and α is the inclination of the perpendicular.

   i. $p = 2$, $\alpha = 30^{o}$ ii. $p = 3\sqrt{3}$, $\alpha = 210^{o}$ iii. $p = 2\sqrt{2}$, $\alpha = 235^{o}$

5. Determine the equation of the straight line which passes through the origin and the mid-point of the line joining $A \equiv (0, -4)$ and $B \equiv (8, 0)$.

6. i. Determine the equation of the straight line which passes through the point (5, 0) and is such that the sum of the x and y intercepts of the line is 9.
   ii. Determine the equation of the straight line which passes through the point (6, -6) and is such that the sum of the x and y intercepts of the line is 5.

7. The coordinates of the foot of the perpendicular from the origin to a straight line is (-2, 2). Determine the equation of the straight line.

8. By applying the concept of a straight line, show that each of the following sets of ordered pairs is collinear.

   i. (3, 0), (-2, -2) and (8, 2)   ii. (1, 3), (3, 5) and (5, 7)

9. Determine the equation of the straight line segment whose two end points are on the two axes, and which is divided in the ratio 1:2 by the point $R \equiv (h, k)$.

10. Show that the equation of the straight line segment whose end points are on the two axes and whose mid-point is $P \equiv (a, b)$, is given $\frac{x}{a} + \frac{y}{b} = 2$.

11. Prove the following geometry results by using only your knowledge on coordinate geometry.
    i. The straight line which joins the mid-points of two sides of a triangle is parallel to the third side and is equal to half its length.
    ii. The distances from the mid-point of the hypotenuse of a right angled triangle to the three vertices are equal.
    iii. If the diagonals of a parallelogram are equal, then it is a rectangle.
    iv. The medians of an equilateral triangle are perpendicular to the sides of the triangle.
    v. The angle in a semi-circle is a right angle.
    vi. If $A\hat{B}C = 90^{o}$ in the right angled triangle ABC, then $AC^2 = AB^2 + BC^2$.

12. Show that the equation of the straight line through the points $(a\cos\alpha, b\sin\alpha)$ and $(a\cos\beta, b\sin\beta)$ is $\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right) + \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right)$

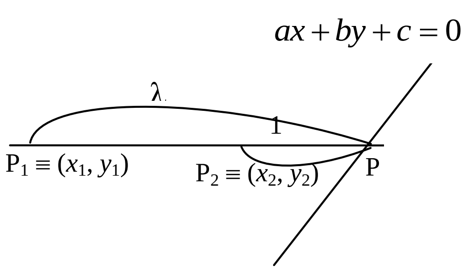
7.13 Determining the Point of Intersection of Two Non-parallel Lines

If $ax+by+c=0$ and $lx+my+n=0$ represent two non-parallel straight lines, and if by simultaneously solving $ax+by+c=0$ and $lx+my+n=0$, the values $x=\alpha$ and $y=\beta$ are obtained, then the intersection point of the two straight lines is (α,β).

7.14 The Location of a Point with respect to a Straight Line

Suppose the straight line joining the points $P_1 \equiv (x_1, y_1)$ and $P_2 \equiv (x_2, y_2)$ is divided into the ratio $\lambda : 1$ by the straight line $ax+by+c=0$.

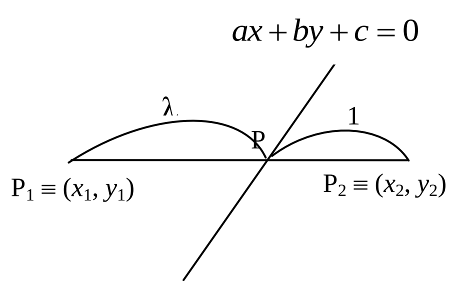
When P_1 and P_2 are on the same side of $ax+by+c=0$



The point P is external to P_1P_2

Then $\lambda < 0$

When P_1 and P_2 are on opposite sides of $ax+by+c=0$



The point P is between P_1 and P_2

Then $\lambda > 0$

Since P $\equiv\left(\left(\dfrac{x_1+\lambda x_2}{1+\lambda}\right),\left(\dfrac{y_1+\lambda y_2}{1+\lambda}\right)\right)$ lies on the line $ax+by+c=0$,

$$a\left(\frac{x_1+\lambda x_2}{1+\lambda}\right)+b\left(\frac{y_1+\lambda y_2}{1+\lambda}\right)+c=0$$

$$a\left(\frac{x_1+\lambda x_2}{1+\lambda}\right)+b\left(\frac{y_1+\lambda y_2}{1+\lambda}\right)+c=0 \Rightarrow a\left(x_1+\lambda x_2\right)+b\left(y_1+\lambda y_2\right)+c\left(1+\lambda\right)=0$$

$$\Rightarrow ax_1+by_1+c+\lambda\left(ax_2+by_2+c\right)=0 \Rightarrow \lambda=-\frac{ax_1+by_1+c}{ax_2+by_2+c}$$

Therefore, if $-\dfrac{ax_1+by_1+c}{ax_2+by_2+c}<0$, P_1 and P_2 both lie on the same side of $ax+by+c=0$.

If $\quad -\dfrac{ax_1+by_1+c}{ax_2+by_2+c} > 0 \quad$ then P_1 and P_2 lie on opposite sides of $ax+by+c=0$.

If ax_1+by_1+c and ax_2+by_2+c have the same sign, then P_1 and P_2 both lie on the same side of $ax+by+c=0$.

If ax_1+by_1+c and ax_2+by_2+c have opposite signs, then P_1 and P_2 lie on opposite sides of $ax+by+c=0$.

From the above we can conclude that all points on one side of the straight line $ax+by+c=0$ are such that $ax+by+c>0$ and all the points on the opposite side of $ax+by+c=0$ are such that $ax+by+c<0$. Therefore, the plane Oxy is divided into three regions by the straight line $ax+by+c=0$.

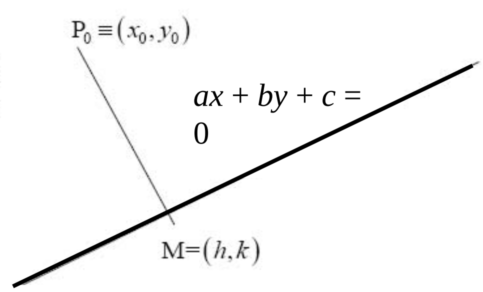
The points on one side of the line are such that $ax+by+c>0$, the points on the other side of the line are such that $ax+by+c<0$ and the points on the line are such that $ax+by+c=0$.

7.15 The Perpendicular Distance from the Point P ≡ (x_0, y_0) to the Straight Line *ax* + *by* + *c* = 0

Since the straight line $ax+by+c=0$ is perpendicular to P_oM
(gradient of $ax+by+c=0$) × (gradient of P_oM) = -1

$$\therefore \left(-\frac{a}{b}\right)\left(\frac{k-y_0}{h-x_0}\right) = -1 \quad \Rightarrow \quad \frac{k-y_0}{h-x_0} = \frac{b}{a}$$

$$\frac{h-x_0}{a} = \frac{k-y_0}{b} = t$$



Then $h-x_0 = at$ and $k-y_0 = bt \Rightarrow h = x_0 + at$ and $k = y_0 + bt$

Since M ≡ (h,k) lies on the line $ax+by+c=0$, we have $ah+bk+c=0$

$$a(x_0+at)+b(y_0+bt)+c=0 \quad \Rightarrow \quad (a^2+b^2)t=-(ax_0+by_0+c)$$

$$\Rightarrow \quad t=-\frac{(ax_0+by_0+c)}{(a^2+b^2)}$$

$$\mathrm{P_0M}^2 = (h-x_0)^2 + (k-y_0)^2 = (at)^2 + (bt)^2$$

$$= (a^2+b^2)t^2 = (a^2+b^2)\left\{-\frac{(ax_0+by_0+c)}{(a^2+b^2)}\right\}^2 = \frac{(ax_0+by_0+c)^2}{(a^2+b^2)}$$

Since P_oM is a length it has to be positive. $\therefore \quad \mathrm{P_0M} = \frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}$

Example

Determine the perpendicular distance from $P_o \equiv (-5, 15)$ to the straight line $3x-4y+25=0$.

Solution

The perpendicular distance from $P_o \equiv (-5, 15)$ to the straight line $3x-4y+25=0$

$$=\left|\frac{3(-5)-4(15)+25}{\sqrt{3^2+4^2}}\right| \quad =\left|\frac{3(-5)-4(15)+25}{\sqrt{9+16}}\right| = \left|\frac{-15-60+25}{5}\right| = \left|\frac{-50}{5}\right| = 10$$

$\therefore$ The perpendicular distance from $P_o \equiv (-5, 15)$ to the straight line $3x-4y+25=0$ = 10 units.

Example

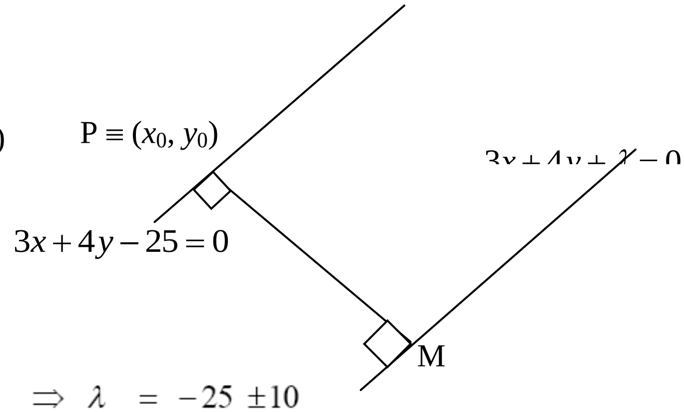
Determine the equations of the two straight lines which are 2 units away from the straight line $3x+4y-25=0$.

Solution

The straight lines which are 2 units away from the line $3x+4y-25=0$ are parallel to it. Therefore, they have the form $3x+4y+\lambda=0$. Since the point $P \equiv (x_0, y_0)$ lies on the straight line, $3x_0+4y_0-25=0$. Therefore, $3x_0+4y_0=25$

Since $P_oM = 2$, $\dfrac{|3x_0 + 4y_0 + \lambda|}{\sqrt{3^2+4^2}} = 2$

$\Rightarrow \dfrac{|25+\lambda|}{\sqrt{3^2+4^2}} = 2$ (since $3x_0 + 4y_0 = 25$)



$\therefore \dfrac{|25+\lambda|}{5} = 2$

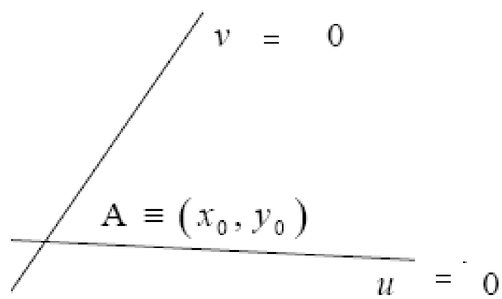
$\Rightarrow |25+\lambda| = 10 \Rightarrow 25+\lambda = \pm 10 \Rightarrow \lambda = -25 \pm 10$

$\therefore \lambda = -15$ or $\lambda = -35$

Therefore, the equations of the two lines which are 2 units away from $3x+4y-25=0$ are $3x+4y-15=0$ and $3x+4y-35=0$

7.16 The Equation of the Straight Line which passes through the Point of Intersection of Two Straight Lines

Suppose $u = 0$ and $v = 0$ are two straight lines which intersect each other.



Suppose $u \equiv ax+by+c=0$ and that $v \equiv lx+my+n=0$

Let A ≡ (x_0, y_0) be the point of intersection of $u = 0$ and $v = 0$.

$$u + \lambda v = 0 \Rightarrow u + \lambda v = ax+by+c+\lambda(lx+my+n) = 0$$

$$\therefore u + \lambda v = (a+\lambda l)x + (b+\lambda m)y + (c+\lambda n) = 0$$

This is a linear equation in x and y. $u+\lambda v=0$ represents a straight line.
Since A ≡ (x_0, y_0) lies on $u = 0$,
$ax_0 + by_0 + c = 0$ ----------------- (1)
Also, since A ≡ (x_0, y_0) lies on $v = 0$,
$lx_0 + my_0 + n = 0$ ----------------- (2)
$(1)+\lambda(2) \Rightarrow ax_0 + by_0 + c + \lambda(lx_0 + my_0 + n) = 0$

This shows that A ≡ (x_0, y_0) lies on the line $u+\lambda v=0$. Therefore, $u+\lambda v=0$ represents the line that passes through the point of intersection A ≡ (x_0, y_0) of the two line $u = 0$ and $v = 0$.

Example

Determine the equation of the straight line that passes through the point of intersection of $3x-y+4=0$ and $x+2y-5=0$ and is

(i) parallel to (ii) perpendicular to
the straight line $2x-5y=0$

Solution

The equation of the straight line through the point of intersection of $3x-y+4=0$ and $x+2y-5=0$ is of the form $3x-y+4+\lambda(x+2y-5)=0$.
$(3+\lambda)x-(1-2\lambda)y+(4-5\lambda)=0$; The gradient of the above line = $\dfrac{3+\lambda}{1-2\lambda}$.
Also, the gradient of $2x-5y=0$ is $\dfrac{2}{5}$.

(i) For $(3+\lambda)x-(1-2\lambda)y+(4-5\lambda)=0$ to be parallel to $2x-5y=0$, we require $\dfrac{3+\lambda}{1-2\lambda}=\dfrac{2}{5}$

$\therefore\ 15+5\lambda=2-4\lambda \Rightarrow 9\lambda=-13 \Rightarrow \lambda=-\dfrac{13}{9}$

Therefore, the equation of the straight line that passes through the point of intersection of $3x-y+4=0$ and $x+2y-5=0$ and is parallel to the straight line $2x-5y=0$ is

$$3x-y+4-\frac{13}{9}(x+2y-5)=0$$
$$\therefore\ 27x-9y+36-13(x+2y-5)=0$$
$$\therefore\ 27x-9y+36-13x-26y+65=0$$
$$\therefore\ 14x-35y+101=0$$

Therefore, the equation of the straight line that passes through the intersection point of $3x-y+4=0$ and $x+2y-5=0$ and is parallel to the straight line $2x-5y=0$ is $14x-35y+101=0$

(ii) The gradient of $2x-5y=0$ is $\dfrac{2}{5}$.

Therefore, the gradient of the straight line perpendicular to $2x-5y=0$ is $-\dfrac{5}{2}$.

For the equation $3x-y+4+\lambda(x+2y-5)=0$ to be perpendicular to $2x-5y=0$, $\dfrac{3+\lambda}{1-2\lambda}=-\dfrac{5}{2}$.

$\therefore\ 2(3+\lambda)=-5(1-2\lambda) \Rightarrow 6+2\lambda=-5+10\lambda$

$\Rightarrow 8\lambda=11 \Rightarrow \lambda=\dfrac{11}{8}$

The straight line that passes through the point of intersection of
$3x-y+4=0$ and $x+2y-5=0$ and is perpendicular to the line $2x-5y=0$ is

$$\therefore\ 3x - y + 4 + \frac{11}{8}\left(x + 2y - 5\right) = 0$$

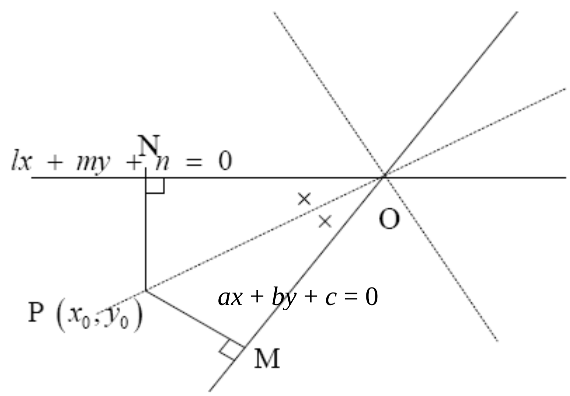
$$\Rightarrow\ 8\left(3x - y + 4\right) + 11\left(x + 2y - 5\right) = 0$$

$$\Rightarrow\ 24x - 8y + 32 + 11x + 22y - 55 = 0 \quad \Rightarrow \quad 35x + 14y - 23 = 0$$

Therefore, the straight line that passes through the point of intersection of
$3x-y+4=0$ and $x+2y-5=0$ and is perpendicular to the line $2x-5y=0$ is
$35x+14y-23=0$.

7.17 The Equation of the Bisector of the Angle between *ax* + *by* + *c* = 0 and *lx* + *my* + *n* = 0

Proof



Let P ≡ (x_0, y_0) be an arbitrary point on either of the bisectors.
Let M and N be the foots of the perpendiculars from P to $ax + by + c = 0$ and $lx + my + n = 0$ respectively.
Then ΔOMP ≡ ΔONP.

$$\therefore\ MP = NP$$

$$\therefore\ \frac{\left|ax_0 + by_0 + c\right|}{\sqrt{a^2+b^2}} = \frac{\left|lx_0 + my_0 + n\right|}{\sqrt{l^2+m^2}}$$

Removing the modulus sign $\dfrac{ax_0 + by_0 + c}{\sqrt{a^2+b^2}} = \pm \dfrac{lx_0 + my_0 + n}{\sqrt{l^2+m^2}}$

Therefore, the locus of P ≡ (x_0, y_0) is $\dfrac{ax + by + c}{\sqrt{a^2+b^2}} = \pm \dfrac{lx + my + n}{\sqrt{l^2+m^2}}$

That is, the equations of the bisectors of the angle between $ax + by + c = 0$ and $lx + my + n = 0$ are given by

$$\frac{ax + by + c}{\sqrt{a^2+b^2}} = \pm \frac{lx + my + n}{\sqrt{l^2+m^2}}$$

The equations of the bisectors are of the form

$$\left(ax + by + c\right) = \pm \sqrt{\frac{a^2+b^2}{l^2+m^2}}\ \left(lx + my + n\right) = 0$$

These equations are of the form, $(ax + by + c) + \lambda\ (lx + my + n) = 0$

$\therefore$ This takes the form of the line that passes through the intersection point of the two lines.

Example

Determine the bisectors of the straight lines $3x - 4y + 2 = 0$ and $12x + 5y - 3 = 0$ and find out which of the two is the bisector of the acute angle between the two lines.

Solution

The bisectors of the two lines $3x - 4y + 2 = 0$ and $12x + 5y - 3 = 0$ are given by

$$\frac{3x-4y+2}{\sqrt{3^2+(-4)^2}} = \pm\frac{12x+5y-3}{\sqrt{12^2+5^2}} \Rightarrow \frac{3x-4y+2}{5} = \pm\frac{12x+5y-3}{13}$$

$$\Rightarrow\ 39x - 52y + 26 = \pm\ (60x + 25y - 15)$$

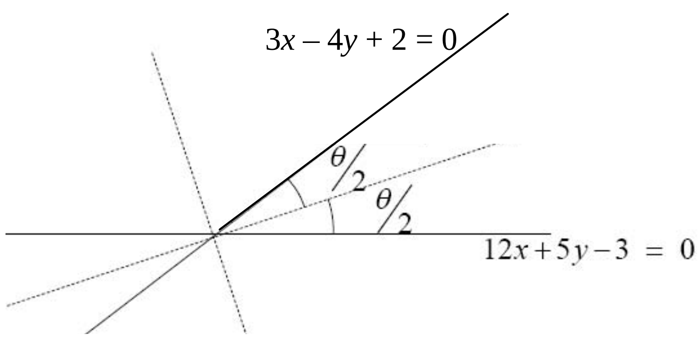
Considering the positive sign:

$$39x - 52y + 26 = 60x + 25y - 15 \Rightarrow\ 21x + 77y - 41 = 0$$

Considering the negative sign

$$39x - 52y + 26 = -60x - 25y + 15 \Rightarrow\ 99x - 27y + 11 = 0$$

Therefore, the two bisectors are $21x + 77y - 41 = 0$ and $99x - 27y + 11 = 0$.



If the acute angle between the two lines is θ, then $\theta < \frac{\pi}{2}$.

Then the acute angle between the bisector and $3x - 4y + 2 = 0$ is $\frac{\theta}{2} < \frac{\pi}{4}$.

$\frac{\theta}{2} < \frac{\pi}{4} \Rightarrow \tan\theta < 1$.

Let us consider the bisector $21x + 77y - 41 = 0$.

The gradient of $21x+77y-41=0$ is $m_1 = -\frac{21}{77} = -\frac{3}{11}$.

The gradient of $3x-4y+2=0$ is $m_2 = \frac{3}{4}$.

Therefore, the tangent of the angle between the bisector $21x+77y-41=0$ and $3x-4y+2=0$ is

$$\left|\frac{m_1-m_2}{1+m_1m_2}\right| = \left|\frac{-\frac{3}{11}-\frac{3}{4}}{1+\left(-\frac{3}{11}\right)\frac{3}{4}}\right| = \left|\frac{-12-33}{44-9}\right| > 1$$

Therefore, the bisector of the acute angle is $99x-27y+11=0$.

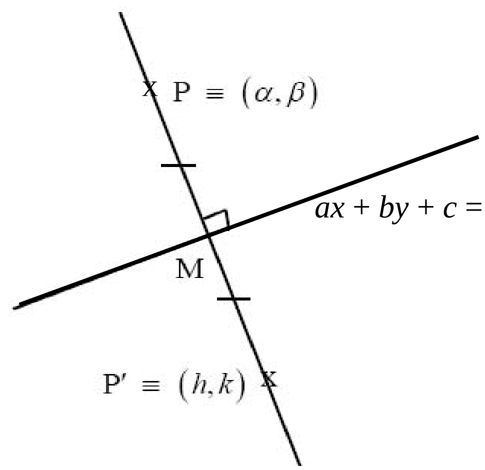
i.e., the bisector of the acute angle between $3x-4y+2=0$ and $12x+5y-3=0$ is $99x-27y+11=0$.

7.18 Determining the Image of P = (α, β) in the straight line *ax* + *by* + *c* = 0.

Let $P' = (h,k)$ be the image of P = (α, β) in $ax + by + c = 0$.

Then the straight line PP′ is perpendicular to $ax + by + c = 0$.

∴ (gradient of PP′) × (gradient of $ax + by + c = 0$) = -1.



$$\therefore \left(\frac{k-\beta}{h-\alpha}\right) \times \left(\frac{-a}{b}\right) = -1$$

$$\frac{k-\beta}{b} = \frac{h-\alpha}{a} = t$$

$$\therefore k-\beta = bt \Rightarrow k = \beta+bt$$

$$\therefore h-\alpha = at \Rightarrow h = \alpha+at$$

Since PM = P′M,

$$\text{M} \equiv \left[\frac{h+\alpha}{2}, \frac{k+\beta}{2}\right] \equiv \left[\frac{\alpha+at+\alpha}{2}, \frac{\beta+bt+\beta}{2}\right]$$

$$\equiv \left[\frac{2\alpha+at}{2}, \frac{2\beta+bt}{2}\right] \equiv \left[\alpha+\frac{at}{2}, \beta+\frac{bt}{2}\right]$$

Since M is a point on $ax + by + c = 0$,

$$a\left(\alpha+\frac{at}{2}\right)+b\left(\beta+\frac{bt}{2}\right)+c=0 \quad .$$

$$\frac{(a^2+b^2)t}{2} = -\ (a\alpha + b\beta + c) \quad , \quad t = -\ \frac{2(a\alpha + b\beta + c)}{(a^2+b^2)}$$

$$\therefore\ \mathrm{P}' \equiv (h, k) \equiv [(\alpha + at),\ (\beta + bt)]$$

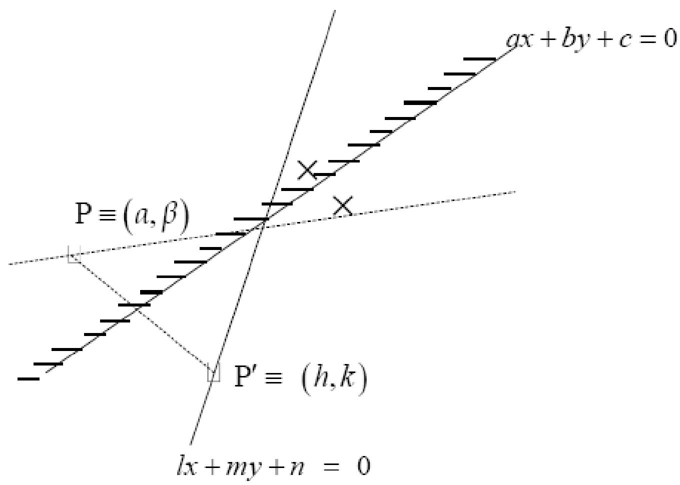
$$\equiv \left[\left(\alpha - \frac{2a(a\alpha + b\beta + c)}{(a^2+b^2)} \right), \quad \left(\beta - \frac{2b\ (a\alpha + b\beta + c)}{(a^2+b^2)} \right) \right]$$

Therefore, the coordinates of the image of P = (α, β) in $ax + by + c = 0$ are

$$\left[\left(\alpha - \frac{2a(a\alpha + b\beta + c)}{(a^2+b^2)} \right), \quad \left(\beta - \frac{2b\ (a\alpha + b\beta + c)}{(a^2+b^2)} \right) \right]$$

7.19 The Equation of the Mirror Image of *lx* + *my* + *n* = 0 in the straight line *ax* + *by* + *c* = 0

Let P = (α, β) be an arbitrary point on the mirror image of $lx + my + n = 0$ in $ax + by + c = 0$. We need to find the mirror image of $lx + my + n = 0$ in $ax + by + c = 0$; i.e., the locus of P = (α, β). For this, we find a relationship between α and β and then replace α and β by x and y respectively.



The image of P = (α, β) in the line $ax + by + c = 0$ should be on the line $lx + my + n = 0$.
Since the image of P = (α, β) in the line $ax + by + c = 0$ is $P' = (h, k)$, by the above section,

$$\mathrm{P}' \equiv \left[\left(\alpha - \frac{2a(a\alpha + b\beta + c)}{(a^2+b^2)} \right), \quad \left(\beta - \frac{2b\ (a\alpha + b\beta + c)}{(a^2+b^2)} \right) \right]$$

is on the line $lx + my + n = 0$.

$$\therefore\ l\left[\alpha - \frac{2a(a\alpha + b\beta + c)}{a^2+b^2}\right] + m\left[\beta - \frac{2b(a\alpha + b\beta + c)}{a^2+b^2}\right] + n = 0$$

$$\therefore\ l\alpha + m\beta + n - \frac{2(al + bm)}{(a^2+b^2)}(a\alpha + b\beta + c) = 0$$

Now by replacing α and β by x and y respectively, we obtain the equation of the mirror image. Therefore, the mirror image of $lx + my + n = 0$ in the line $ax + by + c = 0$, is of the form

$$lx + my + n - \lambda(ax + by + c) = 0$$

Clearly, the mirror image of $lx + my + n = 0$ in the line $ax + by + c = 0$ passes through the point of intersection of the two lines.

Exercises

1. The coordinates of the vertices of a triangle are given. Determine the coordinates of the in-centre.
   (i) (7, -36), (7, 20) and (-8,0) (ii) (8, 12), (0, 6) and (8, 0).

2. Show that the area of the triangle bounded by the lines $y = m_1x + c_1$, $y = m_2x + c_2$ and $x = 0$ is $\dfrac{(c_1 - c_2)^2}{2|m_1 = m_2|}$.

3. Show that the perpendicular distance from the origin to the line joining the points $(\cos\theta, \sin\theta)$ and $(\cos\phi, \sin\phi)$ is $\left|\cos\left(\dfrac{\theta-\phi}{2}\right)\right|$.

4. If the straight lines given by $y = m_1x + c_1$, $y = m_2x + c_2$ and $y = m_3x + c_3$ are concurrent, show that $m_1(c_2 - c_3) + m_2(c_3 - c_2) + m_3(c_1 - c_2)$

5. Find the coordinates of the foot of the perpendicular from (1, 2) to the line $x + 2y - 10 = 0$.

6. Show that the medians of the triangles with the given vertices are concurrent.
   i. (0, 0), (2, 5) and (3, 1) ii. (-1, 1), (3, 10) and (4, 2)

7. Show that the altitudes of the triangles with the given vertices are concurrent.
   i. (0, 5), (1, -1) and (2, 4) ii. (-4, -3), (1, 10) and (5, 5)

8. Determine the angle between the given lines.
   i. $y - 2x = 3$ and $y - 3x = 4$ ii. $x + 2y = 3$ and $2x - 3y = 4$.

9. Determine the equation of the line
   i. through (1, 4) and parallel to $2x + 5y = 7$ ii. through (3, 2) and parallel to $x + 4y = 5$

10. Determine the equation of the line
    i. through (3, 1) and perpendicular to $x - 3y = 7$

    ii. through (5, 3) and perpendicular to $3x - 5y = 11$

11. Determine the angles that the following lines make with $21x - y - 2 = 0$

i. $15x - 8y + 1 = 0$ ii. $12x + 5y - 3 = 0$

12. Determine the distance between the given pairs of parallel lines

i. $4x + 3y = 6, \quad 4x + 3y = 16$ ii. $8x + 15y = 10, \; 8x + 15y = 13$

13. i. Determine the equation of the line which passes through the intersection point of $x + 3y + 1 = 0$ and $2x - y - 5 = 0$ and is parallel to $4x - y = 5$.

ii. Determine the equation of the line which passes through the intersection point of $2x + 3y = 1$ and $3x - 5y = 8$ and is perpendicular to $x - 5y = 8$.

14. Show that the two straight lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are

i. parallel only if $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2}$

ii. perpendicular only if $a_1a_2 = b_1b_2$

15. Show that the straight line perpendicular to the line $x\sec\alpha + y\,\mathrm{cosec}\,\alpha = a$ and through the point $(a\cos^3\alpha, a\sin^3\alpha)$ is $x\cos\alpha - y\sin\alpha = a\cos 2\alpha$

16. Determine the distance between the parallel lines $x + 2y = 5$ and $2x + 4y = 12$.

17. If the straight lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ & $a_3x + b_3y + c_3 = 0$ are concurrent, show that

$a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2) = 0$

Deduce that the lines $x + 2y = 5$, $2x + y = 4$ and $3x - y = 1$ are concurrent.

18. i. Show that the lines $2x + y = 1$, $4x - y = 5$ and $x - 3y = 4$ are concurrent and determine the point of intersection.

ii. Determine the value of *k* such that the lines $x + 3y = 5$, $kx + y = 6$ and $4x - 5y = 3$ are concurrent, and find the point of intersection.

19. A(0, 6), B(4,0) and C(2,2).The line that passes through C and is perpendicular to AB meets the *x*-axis at E. The perpendicular to EA drawn at A meets EC produced at F. Determine

i. the tangent of $E\hat{A}C$ ii. the area of the triangle AEC iii. the coordinates of F.

8. Trigonometry

Competency **:** 15. Solves problems by deriving relationships involving angle measures
16. Interprets circular functions
18. Applies the sine formula and the cosine formula to solve trigonometry problems.

Competency Levels : 15, 16.1, 16.2, 16.3, 18

Subject Content : Angle Measures
Circular Functions
Sine Formula and Cosine Formula

By studying this section you will develop the skills of

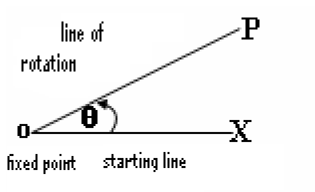
- identifying an angle, a positive angle, a negative angle.
- defining a degree, a radian.
- explaining the relationship between a degree and a radian.
- calculating the length of a circular arc and the area of a sector.
- defining the trigonometric ratios.
- finding the trigonometric ratios of certain angles (0^o, 30^o, 45^o, 60^o, 90^o).
- obtaining the trigonometric ratios of angles such as $-\theta, \dfrac{\pi}{2} \pm \theta, \pi \pm \theta$... etc., in terms of θ.
- determining co-terminal angles.
- identifying the domain and range of trigonometric functions.
- sketching the graphs of trigonometric functions and identifying their periodic nature.
- finding the general solution of trigonometric equations.
- expressing and proving the sine formula and cosine formula for an arbitrary triangle.
- solving problems related to triangles by applying the sine formula and cosine formula.

Introduction

Trigonometry can be considered as an aspect of geometry. Trigonometry is used frequently in the study of measures related to triangles. The Greeks are considered to be the first to study trigonometry. The term trigonometry is made up from the Greek words trigonon (triangle) and metron (measure). The underlying meaning is the "measurement of angles". Trigonometry is applied practically in diverse fields such as space exploration, astronomy, surveys, nautical science, irrigation and architecture, as well as in tasks involving spherical angle measurements. Trigonometry is also used in the study of electricity, electronics, sound waves and magnetic waves. Also, an E.C.G. graph is a trigonometric graph.

8.1 Measurement of Angles

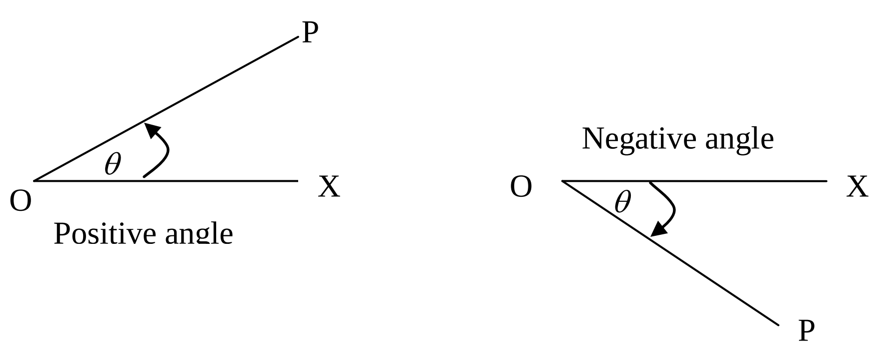
8.1.1 Defining an Angle



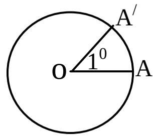
Suppose the line OP initially coincides with the line OX and rotates about the fixed point O. The "amount" that OP has rotated from its initial position is called the angle between OP and OX.

8.1.2 Positive and Negative Angles

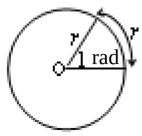
As a convention, when the line of rotation rotates in an anti-clockwise direction, the angle obtained is considered to be a positive angle, and when the line of rotation rotates in a clock-wise direction the angle obtained is considered to be a negative angle.



8.1.3 The Degree and the Radian

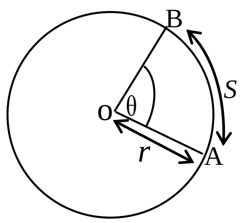


Degree: The angle between two radii of a circle, which cut off an arc of length $\frac{1}{360}$ of the circumference, is defined as 1^{o}.



Radian: The angle between two radii of a circle which cut off an arc of length the radius of the circle is defined as 1 radian

8.1.4 The Length of an Arc of a Circle



Radius – r
The angle subtended at the centre by the arc AB - θ
The length of the arc AB – S

According to the definition of the radian; $\theta = \dfrac{S}{r}$.

$$\boxed{\boldsymbol{S = r\theta}}$$

8.1.5 The Relationship between the Degree and the Radian

The circumference of a circle of radius r = $2\pi r$
The angle subtended at the centre by the circumference = 360^{o}
According to the definition of the radian,

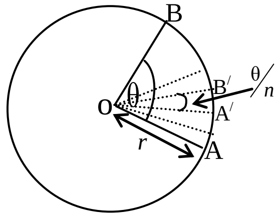
the angle that the circumference of a circle subtends at the centre = $\dfrac{2\pi r}{r}$ = 2π radians

Accordingly, 2π radians = 360^{o}

$$\boxed{\boldsymbol{\pi \text{ radians} = 180^{o}}}$$

π is an irrational number $\dfrac{22}{7}$ is only an approximate value for π

8.1.6 The Area of a Sector of a Circle



Let us consider the sector AOB of a circle of radius r. If this is divided into n very small sectors and one of them is taken as A′OB′, then

$A'\hat{O}B' = \dfrac{\theta}{n}$, the length of the arc A′B′ = $r\dfrac{\theta}{n}$.

Since A′OB′ is very small, we can consider it as a triangle, of which the altitude is the radius r.

Accordingly,

the area of A′OB′ = $\frac{1}{2}\times \text{A}'\text{B}'\times r = \frac{1}{2}\times\left(r\times\frac{\theta}{n}\right)\times r = \frac{1}{2}r^2\frac{\theta}{n}$

Since the sector AOB is made up of n such triangles,

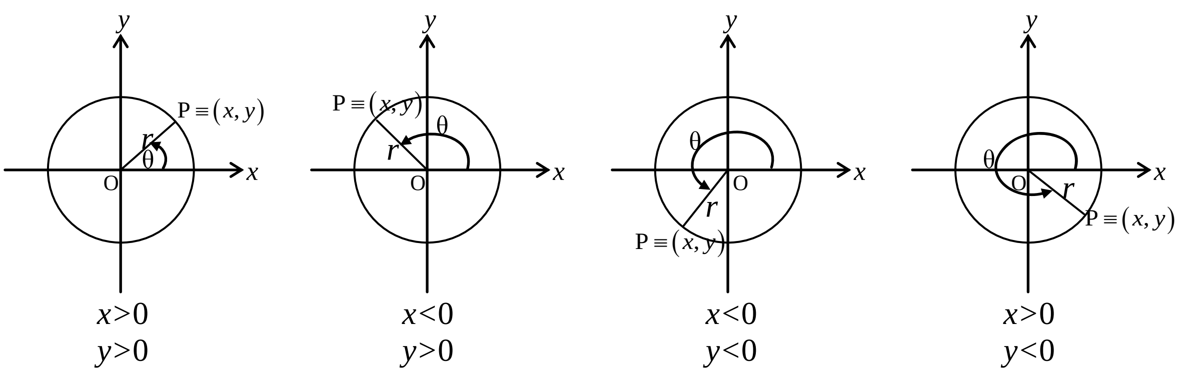
the area of AOB = n × area of A′OB′ = $n\left(\frac{1}{2}r^2\frac{\theta}{n}\right) = \frac{1}{2}r^2\theta$

The area of the sector AOB = $\boldsymbol{\frac{1}{2}r^2\theta}$

8.2 Circular Functions

8.2.1 Trigonometric Functions

The following figures represents the position of the point P in the four quadrants of the Oxy coordinate plane, when the starting line is taken as the positive x-axis, and OP is rotated in an anti-clockwise direction such that $x\hat{\text{O}}\text{P} = \theta$. Here, OP = r and P ≡ (x, y).



For each of the above four cases, six trigonometric ratios, namely,

$\sin\theta = \frac{y}{r},\quad \cos\theta = \frac{x}{r},\quad \tan\theta = \frac{y}{x},\quad \operatorname{cosec}\theta = \frac{r}{y},\quad \sec\theta = \frac{r}{x},\quad \cot\theta = \frac{x}{y}$

are defined.

8.2.2 The Sign of the Trigonometric Ratios in Each Quadrant

Let OP = $r > 0$.

When OP is in the first quadrant, $x > 0$ and $y > 0$.

$$\therefore\ \sin\theta = \frac{y}{r} > 0,\ \cos\theta = \frac{x}{r} > 0,\ \tan\theta = \frac{y}{x} > 0,\ \operatorname{cosec}\theta = \frac{r}{y} > 0,\ \sec\theta = \frac{r}{x} > 0,\ \cot\theta = \frac{x}{y} > 0$$

When OP is in the second quadrant, $x < 0$ and $y > 0$.

$$\therefore\ \sin\theta = \frac{y}{r} > 0,\ \cos\theta = \frac{x}{r} < 0,\ \tan\theta = \frac{y}{x} < 0,\ \operatorname{cosec}\theta = \frac{r}{y} > 0,\ \sec\theta = \frac{r}{x} < 0,\ \cot\theta = \frac{x}{y} < 0$$

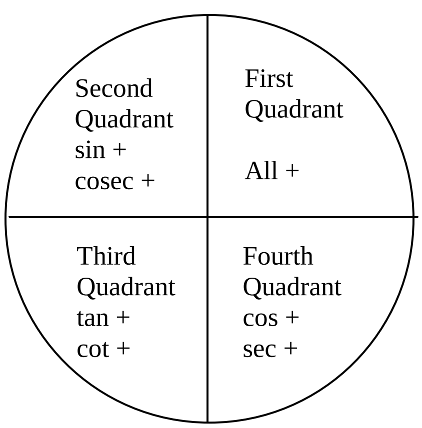
When OP is in the third quadrant, $x < 0$ and $y < 0$.

$$\therefore\ \sin\theta = \frac{y}{r} < 0,\ \cos\theta = \frac{x}{r} < 0,\ \tan\theta = \frac{y}{x} > 0,\ \operatorname{cosec}\theta = \frac{r}{y} < 0,\ \sec\theta = \frac{r}{x} < 0,\ \cot\theta = \frac{x}{y} > 0$$

When OP is in the fourth quadrant, $x > 0$ and $y < 0$.

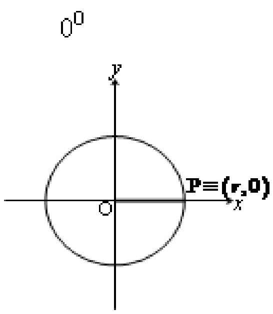
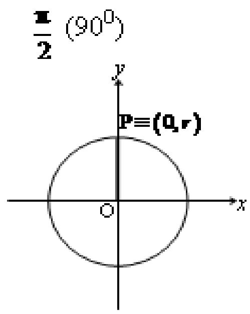
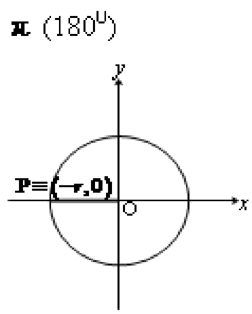
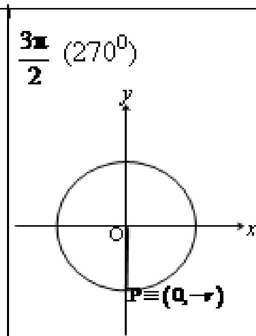
$$\therefore\ \sin\theta = \frac{y}{r} < 0,\ \cos\theta = \frac{x}{r} > 0,\ \tan\theta = \frac{y}{x} < 0,\ \operatorname{cosec}\theta = \frac{r}{y} < 0,\ \sec\theta = \frac{r}{x} > 0,\ \cot\theta = \frac{x}{y} < 0$$

This can be summarized as follows:



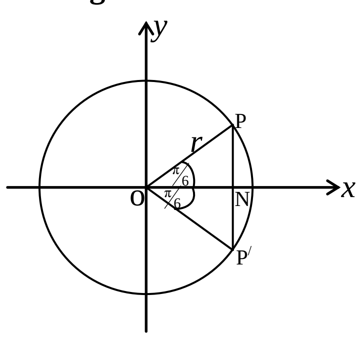
8.2.3 The Trigonometric Ratios of Several Angles

When the length of OP is r

			
Here P ≡ (r, 0)	Here P ≡ (0, r)	Here P ≡ (-r, 0)	Here P ≡ (0,- r)
$\sin 0^o = \frac{0}{r} = 0$	$\sin\frac{\pi}{2} = \frac{r}{r} = 1$	$\sin\pi = \frac{0}{r} = 0$	$\sin\frac{3\pi}{2} = \frac{-r}{r} = -1$
$\cos 0^o = \frac{r}{r} = 1$	$\cos\frac{\pi}{2} = \frac{0}{r} = 0$	$\cos\pi = \frac{-r}{r} = -1$	$\cos\frac{3\pi}{2} = \frac{0}{r} = 0$
$\tan 0^o = \frac{0}{r} = 0$	tan $\frac{\pi}{2}$ is undefined	$\tan\pi = \frac{0}{-r} = 0$	tan $\frac{3\pi}{2}$ is undefined
cosec 0^o is undefined	$\text{cosec}\frac{\pi}{2} = \frac{r}{r} = 1$	cosec π is undefined	$\text{cosec}\frac{3\pi}{2} = \frac{r}{-r} = -1$
$\sec 0^o = \frac{r}{r} = 1$	sec $\frac{\pi}{2}$ is undefined	$\sec\pi = \frac{r}{-r} = -1$	sec $\frac{\pi}{2}$ is undefined
cot 0^o is undefined	$\cot\frac{\pi}{2} = \frac{0}{r} = 0$	cot π is undefined	$\cot\frac{3\pi}{2} = \frac{0}{-r} = 0$

The trigonometric ratios for the above four cases can be easily obtained. This is because the x and y coordinates of the point P could be identified easily. However, when P is a point on the circle different to the above positions, the x and y coordinates of P relevant to the corresponding angle have to be determined first.

8.2.4 The Trigonometric Ratio of the Angle $\frac{\pi}{6}(30^o)$



$N\hat{O}P = \frac{\pi}{6}$

Let OP = r

Let us consider the point P′ on the circle such that $N\hat{O}P' = \frac{\pi}{6}$**,** as indicated in the figure.

OPP′ is an equilateral triangle. [since OP = OP′ = r and $P\hat{O}P' = \frac{\pi}{3}(60^{\circ})$].

Therefore, PP′ = r.

Since Δ OPN ≡ Δ ONP′, PN = $\frac{r}{2}$.

By applying Pythagoras' Theorem to the right angled triangle OPN we obtain,

$ON^2 + NP^2 = OP^2$.

$ON^2 + \frac{r^2}{4} = r^2$.

$ON^2 = \frac{3r^2}{4}$.

$ON = \frac{\sqrt{3}r}{2}$. $\qquad\qquad \therefore\ P \equiv \left[\frac{\sqrt{3}r}{2}, \frac{r}{2}\right]$.

Accordingly,

$$\sin\frac{\pi}{6} = \frac{\frac{r}{2}}{r} = \frac{1}{2} \qquad\qquad \operatorname{cosec}\frac{\pi}{6} = 2$$

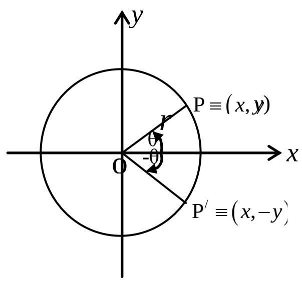
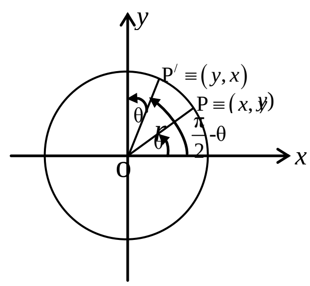
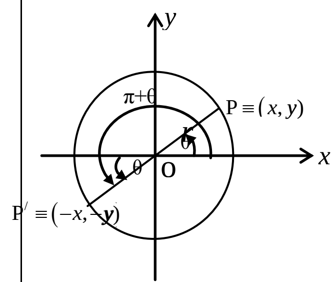
$$\cos\frac{\pi}{6} = \frac{\frac{\sqrt{3}r}{2}}{r} = \frac{\sqrt{3}}{2} \qquad\qquad \sec\frac{\pi}{6} = \frac{2}{\sqrt{3}}$$

$$\tan\frac{\pi}{6} = \frac{\frac{r}{2}}{\frac{\sqrt{3}r}{2}} = \frac{1}{\sqrt{3}} \qquad\qquad \cot\frac{\pi}{6} = \sqrt{3}$$

In the same manner, by finding the coordinates of P corresponding to the angles $\frac{\pi}{4}$ and $\frac{\pi}{3}$, the trigonometric ratios of these angles can be obtained.

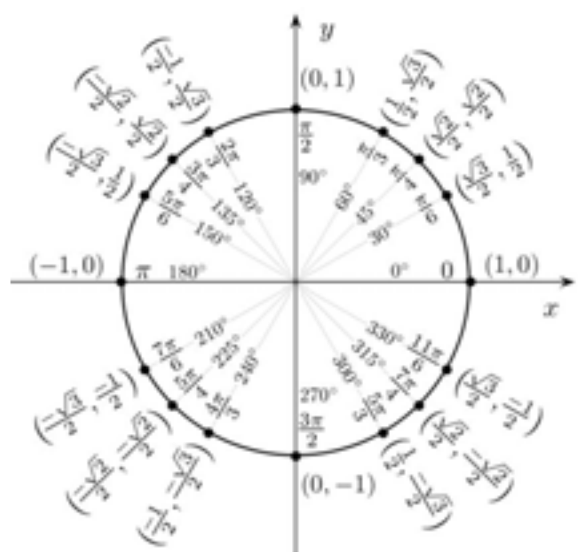
Now, let us find the trigonometric ratios corresponding to an arbitrary angle θ obtained by rotating OP about the point O in either a clockwise direction or an anti-clockwise direction.

8.2.5 Determining the Trigonometric Ratios of $-\theta$, $\frac{\pi}{2}\pm\theta$, $\pi\pm\theta$, ... etc, in terms of θ.

$-\theta$	$\frac{\pi}{2}-\theta$	$\pi+\theta$
		
$\sin(-\theta)=\frac{-y}{r}=-\sin\theta$	$\sin\left(\frac{\pi}{2}-\theta\right)=\frac{x}{r}=\cos\theta$	$\sin(\pi+\theta)=\frac{-y}{r}=-\sin\theta$
$\cos(-\theta)=\frac{x}{r}=\cos\theta$	$\cos\left(\frac{\pi}{2}-\theta\right)=\frac{y}{r}=\sin\theta$	$\cos(\pi+\theta)=\frac{-x}{r}=-\cos\theta$
$\tan(-\theta)=\frac{-y}{x}=-\tan\theta$	$\tan\left(\frac{\pi}{2}-\theta\right)=\frac{x}{y}=\cot\theta$	$\tan(\pi+\theta)=\frac{-y}{-x}=\tan\theta$
$\mathrm{cosec}(-\theta)=-\mathrm{cosec}\theta$	$\mathrm{cosec}\left(\frac{\pi}{2}-\theta\right)=\sec\theta$	$\mathrm{cosec}(\pi+\theta)=-\mathrm{cosec}\theta$
$\sec(-\theta)=\sec\theta$	$\sec\left(\frac{\pi}{2}-\theta\right)=\mathrm{cosec}\theta$	$\sec(\pi+\theta)=-\sec\theta$
$\cot(-\theta)=-\cot\theta$	$\cot\left(\frac{\pi}{2}-\theta\right)=\tan\theta$	$\cot(\pi+\theta)=\cot\theta$

By obtaining the coordinates of the point P′ in terms of the x, y coordinates of the point P, the trigonometric ratios of angles such as $\frac{\pi}{2}+\theta$, $\pi-\theta$, $\frac{3\pi}{2}-\theta$ can also be obtained.

Study the given diagram of a circle, containing the coordinates of the points on the circle corresponding to the angles $\frac{2\pi}{3}$, $\frac{3\pi}{4}$, $\frac{5\pi}{6}$, $\frac{7\pi}{6}$, $\frac{5\pi}{4}$, $\frac{4\pi}{3}$, $\frac{5\pi}{3}$, $\frac{7\pi}{4}$ and $\frac{11\pi}{6}$, obtained using the above principle.



Complete the following table by using the diagram given above

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\operatorname{cosec}\theta$	$\sec\theta$	$\cot\theta$
$0=(0^0)$						
$\frac{\pi}{6}=(30^0)$						
$\frac{\pi}{4}=(45^0)$						
$\frac{\pi}{3}=(60^0)$						
$\frac{\pi}{2}=(90^0)$						
$\frac{2\pi}{3}=(120^0)$						
$\frac{3\pi}{4}=(135^0)$						
$\frac{5\pi}{6}=(150^0)$						
$\pi=180^0$						
$\frac{7\pi}{6}=(210^0)$						
$\frac{5\pi}{4}=(225^0)$						
$\frac{4\pi}{3}=(240^0)$						
$\frac{3\pi}{2}=(270^0)$						
$\frac{5\pi}{3}=(300^0)$						
$\frac{7\pi}{4}=(315^0)$						
$\frac{11\pi}{6}=(330^0)$						

By observing the table and the diagram, propose an easy method to find the trigonometric ratios of angles which do not lie between 0 and $\frac{\pi}{2}$.

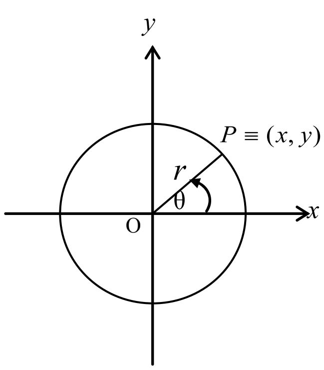
8.2.6 Co-terminal Angles

If we increase the magnitude of the angle determined by OP by a multiple of 2π, OP rotates once or several times about O and returns to the original position. The angle obtained is $2n\pi+\theta; \quad n \in \mathbb{Z}$, when OP, the line of rotation corresponding to the angle θ, is rotated n full circles about O. In such situations, θ and $2n\pi+\theta; \quad n \in \mathbb{Z}$ are called co-terminal angles.
$2n\pi+\theta$ has the same trigonometric ratios as θ.

Example:
Consider the angle $\frac{7\pi}{3}$.
When this is written in the form $2n\pi+\theta$, we obtain $\frac{7\pi}{3} = 2\pi + \frac{\pi}{3}$.
Therefore, $\frac{7\pi}{3}$ and $\frac{\pi}{3}$ are co-terminal angles.

8.3 Graphs of Circular Functions

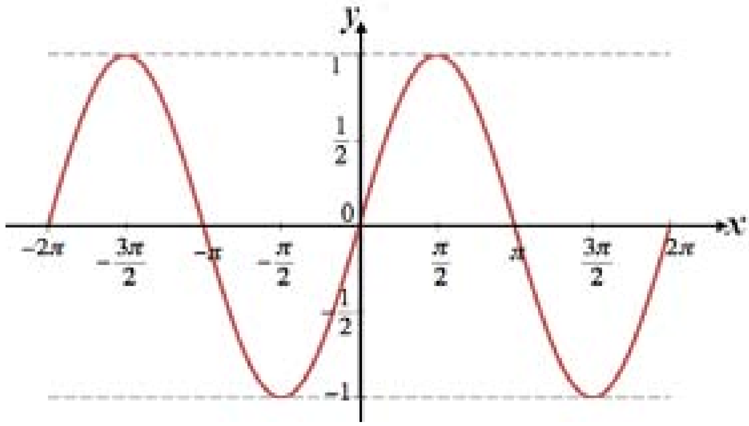
8.3.1 The Domain and the Range of Trigonometric Functions



Function	Domain	Range
$y=\sin\theta$	$\mathbb{R}$	[-1,1]
$y=\cos\theta$	$\mathbb{R}$	[-1,1]
$y=\tan\theta$	$\mathbb{R} - \{(2n+1)\frac{\pi}{2}; n \in Z\}$	$\mathbb{R}$
$y=\text{cosec}\,\theta$	$\mathbb{R} - \{n\pi : n \in Z\}$	$\mathbb{R}$ – (-1,1)
$y=\sec\theta$	$\mathbb{R} - \{(2n+1)\frac{\pi}{2}; n \in Z\}$	$\mathbb{R}$ – (-1,1)
$y=\cot\theta$	$\mathbb{R} - \{n\pi : n \in Z\}$	$\mathbb{R}$

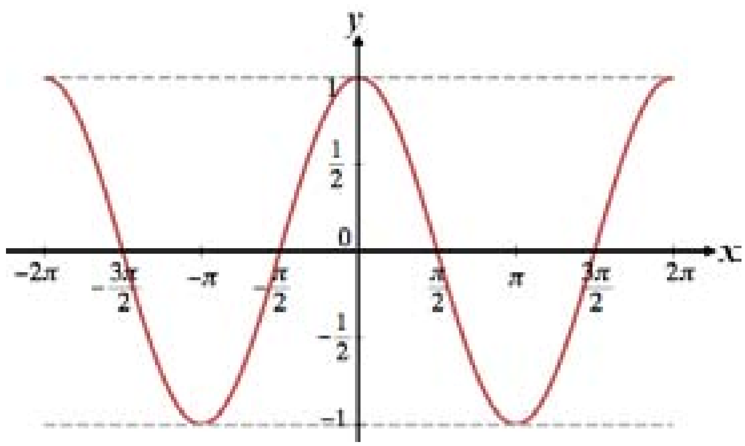
8.3.2 Graphs of Trigonometric Functions and their Periodic Nature

The graph of $y = \sin x$



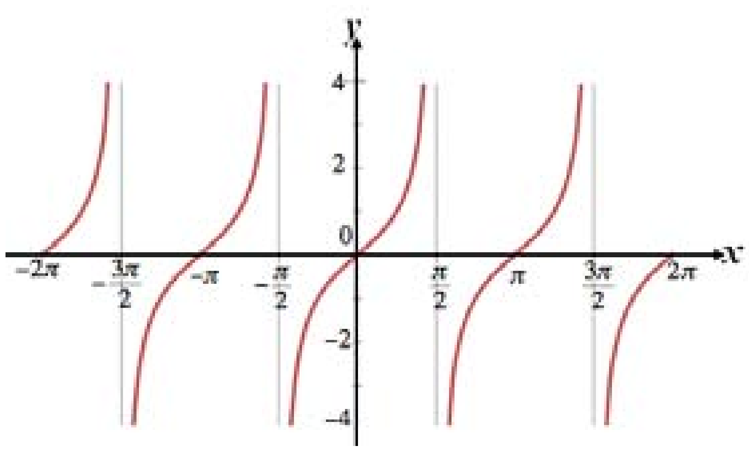
The graph reveals a periodic nature, with the same pattern being repeated in intervals of length 2π

The graph of $y = \cos x$



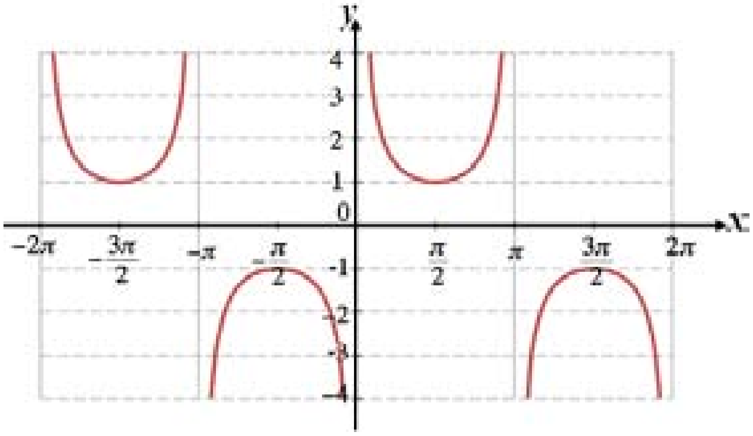
The graph reveals a periodic nature, with the same pattern being repeated in intervals of length 2π

The graph of $y = \tan x$



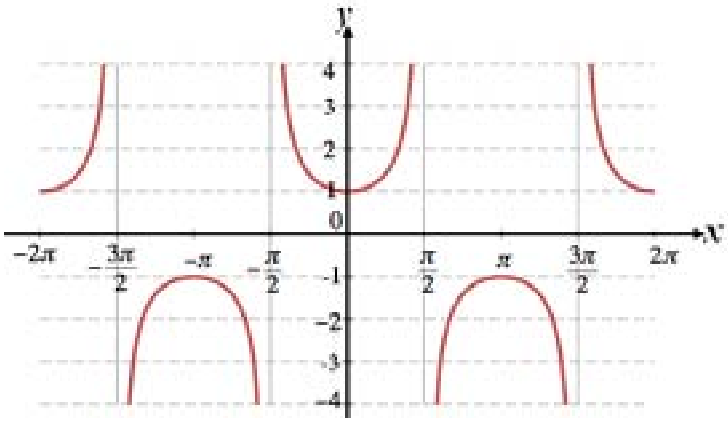
The graph reveals a periodic nature, with the same pattern being repeated in intervals of length π

The graph of $y = \text{cosec}x$



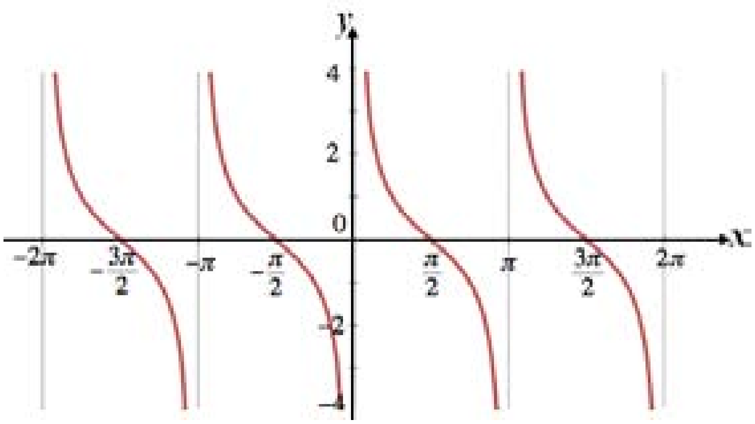
The graph reveals a periodic nature, with the same pattern being repeated in intervals of length 2π

The graph of $y = \sec x$



The graph reveals a periodic nature, with the same pattern being repeated in intervals of length 2π

The graph of $y = \cot x$



The graph reveals a periodic nature, with the same pattern being repeated in intervals of length π

Find Out

$$\left.\begin{array}{l} y = \sin x + c \\ y = \cos x + c \\ y = \tan x + c \end{array}\right\}$$
The graphs of these functions are obtained by translating the graphs of $y = \sin x$, $y = \cos x$, $y = \tan x$ respectively by an amount c along the y-axis.

When $\alpha > 0$,

$$\left.\begin{array}{l} y = \sin(x + \alpha) \\ y = \cos(x + \alpha) \\ y = \tan(x + \alpha) \end{array}\right\}$$
The graphs of these functions are obtained by translating the graphs of $y = \sin x$, $y = \cos x$, $y = \tan x$ respectively by an amount $-\alpha$ along the x-axis.

Similarly,

$$\left.\begin{array}{l} y = \sin(x - \alpha) \\ y = \cos(x - \alpha) \\ y = \tan(x - \alpha) \end{array}\right\}$$
The graphs of these functions are obtained by translating the graphs of $y = \sin x$, $y = \cos x$, $y = \tan x$ respectively by an amount α along the x-axis.

When k is a constant,

$$\left.\begin{array}{l} y = \sin kx \\ y = \cos kx \\ y = \tan kx \end{array}\right\}$$
The graphs of these functions are obtained by translating the arbitrary point (λ, μ) on the graphs of $y = \sin x$, $y = \cos x$, $y = \tan x$ respectively to the point $(\frac{\lambda}{k}, \mu)$.

Similarly,

$$\left.\begin{array}{l} y = k\sin x \\ y = k\cos x \\ y = k\tan x \end{array}\right\}$$
The graphs of these functions are obtained by translating the arbitrary point (λ, μ) on the graphs of $y = \sin x$, $y = \cos x$, $y = \tan x$ respectively to the point $(\lambda, k\mu)$.

8.4 The General Solution of Trigonometric Equations

The general solution of $\sin x = \sin \alpha$ is

$$\boxed{x = n\pi + (-1)^n \alpha\ ; \quad n \in \mathbb{Z}}$$

The general solution of $\cos x = \cos \alpha$ is

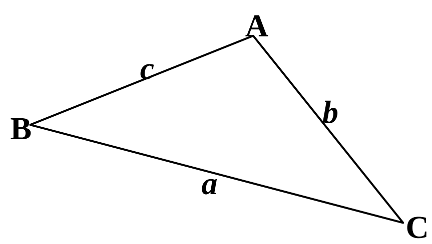
$$\boxed{x = 2n\pi \pm \alpha\ ; \quad n \in \mathbb{Z}}$$

The general solution of $\tan x = \tan \alpha$ is

$$\boxed{x = n\pi + \alpha\ ; \quad n \in \mathbb{Z}}$$

8.5 The sine Formula (sine Rule) and the cosine Formula (cosine Rule)

For any triangle ABC



The sine Formula

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

The cosine Formula

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

It can be shown that these two formulae are true for any triangle ABC.

Exercises

1. Express each of the following angles in radians.

   (*i*) 45^0 (*ii*) 60^0 (*iii*) -135^0 (*iv*) 396^0 (*v*) -30^0

2. Express each of the following angles in degrees.

   (*i*) $\frac{5\pi}{3}$ (*ii*) $\frac{11\pi}{6}$ (*iii*) $\frac{-\pi}{12}$ (*iv*) 0.19π (*v*) $\frac{3\pi}{4}$

3. Find the angle subtended on the centre of a circle of radius 12cm by an arc of length 24cm, in degrees and in radians.

4. In a calibrated circle of diameter 49cm, there are 5 numbers at equal distance from each other. Find the angle subtended at the centre by an arc between two consecutive numbers. Thereby find the length of the arc.

5. The radius of the wheel of a vehicle is 25cm. Find the angle that the wheel has rotated when the vehicle has traveled a distance of 11cm.

6 An angle of 30^o is subtended at the centre by the arc joining two points A and B on the circumference of a circular lamina of radius 3.5cm and centre O. Calculate the length of the arc AB and the area of the sector AOB.

7 The length of the longest chord of a circle of centre O is 28cm. What is the magnitude in degrees, of the angle subtended at the centre of the circle, by a sector of area 231cm^2?

8 Examine whether the following relationships are true or not.

(i) $\cos 60^0 = 2\cos^2 30^0 - 1$

(ii) $\sin 30^0 = \sqrt{\dfrac{1-\cos 60^0}{2}}$

(iii) $\sin^2 30^0 + \sin^2 45^0 + \sin^2 60^0 = \dfrac{3}{2}$

(iv) $\cos 45^0 \cos 60^0 - \sin 45^0 \sin 60^0 = \dfrac{-\left(\sqrt{3}-1\right)}{2\sqrt{2}}$

(v) $4\cot^2 45^0 - \sec^2 60^0 + \sin^2 30^0 = \dfrac{1}{4}$

(vi) $\tan 60^0 = \dfrac{2\tan 30^0}{1-\tan^2 30^0}$

(vii) $\sin 60^0 = 2\sin 30^0 \cos 30^0$

($viii$) $1+\tan^2 30^0 = \sec^2 30^0$

(ix) $\cos^2 45^0 - \sin^2 45^0 = \cos 90^0$

(x) $\operatorname{cosec}^2 45^0 \sec^2 30^0 \sin^2 90^0 \cos 60^0 = 1\dfrac{1}{3}$

9. Determine the six trigonometric ratios of each of the following angles.

(i) 210^0 (ii) -180^0 (iii) $\dfrac{17\pi}{6}$ (iv) $\dfrac{-3\pi}{4}$ (v) 300^0

(vi) 750^0 (vii) -1020^0 ($viii$) -720^0 (ix) $\dfrac{11\pi}{3}$ (x) 450^0

10. Show that $\cos A + \sin(270^0 + A) - \sin(270^0 - A) + \cos(180^0 + A) = 0$.

11. Sketch the graph of each of the following functions within the range $-360^0 \le \theta \le 360^0$.

(i) $y = 1+\sin\theta$ (ii) $y = \sec 2\theta$ (iii) $f(\theta) = \tan(\theta + 90^0)$

$(iv)\ f(\theta) = 2\sin\theta$ $(iv)\ y = 1 + \text{cosec}(2\theta\text{-}180^0)$

12. If $8\left(\cos^6 x + \sin^6 x\right) = 5 + 3\cos 4x$, sketch the graph of $y = \cos^6 x + \sin^6 x$ within the range $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

13. Determine the general solution of each of the following equations.

$(i)\ \tan\theta = \frac{1}{\sqrt{3}}$ $(ii)\ \sin\theta = -\frac{1}{2}$ $(iii)\ \sec^2\theta = \frac{4}{3}$

$(iv)\ 2\cot^2\theta = \text{cosec}^2\theta$ $(v)\ \sin^2\theta = 1$

14. Determine the solution of each of the following equations for $0 \le \theta \le 360^0$.

$(i)\ \cos^2\theta = \frac{1}{2}$ $(ii)\ \sqrt{3}\cos^2\theta + \left(1\text{-}\sqrt{3}\right)\sin\theta\cos\theta\text{-}\sin^2\theta = 0$

$(iii)\ \text{cosec}^2\theta = \frac{4}{3}$ $(iv)\ 4\sin^2\theta + 1 = 3\text{cosec}^2\theta$

$(v)\ 2\sin\theta\cos\theta\text{-}\sin\theta = 0$

15. Determine the solution of $\sin^2 2\alpha = \frac{1}{2}$ in terms of α, for $-360^0 \le 2\alpha \le 360^0$.

16. Determine the solution of $2\cos^2\theta - \cos\theta\text{-}1 = 0$ for $-180^0 \le \theta \le 180^0$.

17. Determine the solution of $\sin\theta = \frac{\sqrt{3}}{2}$, graphically.

18. Given that $a = \sqrt{3},\ b = \sqrt{2},\ c = \frac{\sqrt{6}+\sqrt{2}}{2}$ for a triangle ABC in the usual notation, determine $\hat{A},\ \hat{B},\ \hat{C}$.

19. The foot of the perpendicular from the vertex A to the side BC of the triangle ABC in the usual notation is D. Show that $a = b\cos C + c\cos B$.

20. Show that $a\left[b\cos C - c\cos B\right] = b^2 - c^2$ for a triangle ABC in the usual notation.

21. If $\hat{A} : \hat{B} : \hat{C}$ equals $1:2:3$ for a triangle ABC in the usual notation, prove using the sine rule that the corresponding sides of the triangle are in the ratio $1:\sqrt{3}:2$.

22. Express the cosine rule for a triangle ABC in the usual notation and prove it. For the triangle ABC in the usual notation, show that $\frac{1}{a+c}+\frac{1}{b+c}=\frac{3}{a+b+c}$ if and only if $\hat{C}=\frac{\pi}{3}$.

(G.C.E. Advanced level – 1997 and 2010)

23. For a triangle ABC with sides of length a, b and c in the usual notation, it is given that $\frac{b+c}{2k-1}=\frac{c+a}{2k}=\frac{a+b}{2k+1}$. Here k is an integer greater than 2 and not equal to 4.

Show that $\frac{\sin A}{k+1}=\frac{\sin B}{k}=\frac{\sin C}{k-1}$ and that $\frac{\cos A}{(k-4)(k+1)}=\frac{\cos B}{k^2+2}=\frac{\cos C}{(k+4)(k-1)}$.

(G.C.E. Advanced level – 2000)