Geometry: Basics and Classics

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Many areas of pre-calculus math are only briefly covered in the standard high school curriculum despite the vastness of the topic, but geometry is the best example of this. Many students learn the basics, leaving them with only a glimpse into the world that is Euclidean geometry. I hope by the end of today, all of you will have a feeling for the vastness of this topic, and at least have a start to solving problems from this world.

Notes: Due to difficulties I have experienced trying to put geometric diagrams into LaTeX documents, and to save space, no geometric diagrams appear below. I will attempt to put all relavent diagrams on the board during the lecture. For similar reasons, where some textbooks use the notation $\overline{AB}$, we will always just say AB to mean the line, segment, or length of the segment.

We will begin where the curriculum often ends, with cyclic quadrilaterals.

1 Cyclic quadrilaterals

A *cyclic quadrilateral*, also known as an inscribed quadrilateral, is a quadrilateral that is inscribed in a circle. That is, if A, B, C, and D lie on a circle in that order, then $ABCD$ is a cyclic quadrilateral. There are many things to prove with such quadrilaterals, usually requiring knowledge of inscribed angle theorems. Here are four major results:

1. If X is the intersection of diagonals AC and BD, triangles AXD and BXC are similar.
2. A quadrilateral $ABCD$ is cyclic if and only if $\angle DAB + \angle BCD = 180°$. (CQ-1)
3. (Ptolemy's Theorem) In cyclic quadrilateral $ABCD$, $AC \cdot BD = AB \cdot CD + BC \cdot DA$. Moreover, Ptolemy's Inequality says that $AC \cdot BD > AB \cdot CD + BC \cdot DA$ if $ABCD$ is not a cyclic quadrilateral.
4. (Brahmagupta's Formula) If the lengths of the sides of a cyclic quadrilateral are a, b, c, d and the semiperimeter $s = \frac{a+b+c+d}{2}$, then the area of the cyclic quadrilateral is equal to $\sqrt{(s-a)(s-b)(s-c)(s-d)}$. Note that Heron's formula for the area of a triangle is a special case of this.

To prove the first, we simply note that, $\angle DAX = \angle XBC$ because they are both inscribed angles subtending the same arc. Likewise, $\angle ADX = \angle XCB$, so $\triangle DAX \sim \triangle XBC$, as desired. The proof of the second result is left to the reader.[1]

Ptolemy's Theorem has several very different proofs, but we will prove it when we get to Simson lines later. For now, accept Brahmagupta's Theorem as true. The proof is too long to write out here, and you can read about it in *Geometry Revisited* by Coxeter and Greitzer. It is not surprisingly similar to the proof of Heron's formula, if you are familiar with that.

[1] In order to organize the various problems left to the reader in this handout, denote this as "CQ-1."

Here are a couple more examples of cyclic quadrilaterals in action:
CQ-2. From an arbitrary point M on leg BC of right triangle ABC perpendicular MN is dropped on hypothenuse AB. Prove that $\angle MAN = \angle MCN$.
CQ-3. Prove that a trapezoid is cyclic if and only if it is isosceles.

2 Triangles

Much if not most of geometry takes place in the triangle, the simplest polygon. In generalized notation, we use triangle ABC, with side lengths $BC = a, CA = b, AB = c$, area $[ABC]$, altitudes AD, BE, CF (with lengths h_a, h_b, h_c) medians AA', BB', CC' (lengths m_a, m_b, m_c), angle bisectors AA_1, BB_1, CC_1 (lengths l_a, l_b, l_c), orthocenter H, centroid G, incenter I, circumcenter O, excenters I_A, I_B, I_C, and generalized cevians AX, BY, and CZ. We discuss all of these parts of a triangle below.

Note: We assume knowledge of the Laws of Sines and Cosines and Heron's Formula. Basic terminology of homothecies (or similarity transformations) will be provided in the lecture.

2.1 Cevians and Centers

A *cevian* is a line from one vertex to the opposite side (or an extension thereof). Examples include the *altitudes*, *medians*, and *angle bisectors*.

2.1.1 Altitudes and the Orthocenter

An altitude AD in triangle ABC is a segment extending from vertex A perpendicular to line BC. It is easy to see that if either angle at B or C is obtuse, AD lies outside the triangle; otherwise it is within it.

Altitudes are most useful when computing areas, given by the formula $[ABC] = ah_a = bh_b = ch_c$. One major result is that each altitude length is inversely proportional to its corresponding side length, which is easy to see. A notable special case is the altitude to the hypotenuse AB of a right triangle, which has length $h_c = \frac{ab}{c}$.

The altitudes all intersect at the *orthocenter* H, a result we will prove with Ceva's Theorem. This gives rise to cyclic quadrilaterals such as $AEHF$.

The bases of the altitudes form the *orthic triangle* DEF. One classic problem states that the altitudes of the original triangle bisect the angles of the orthic triangle, making the original orthocenter the incenter of DEF, as we'll see later. In fact, that is our first problem relating to altitudes, known here as A-1.

Most problems involving altitudes require no more complicated methods than angle chasing and cyclic quadrilaterals reviewed above. All of the following problems use the notation introduced above. Here they are:
A-2. (AIME II 2006/15)[2] Given that x, y and z are real numbers that satisfy:

$$\begin{aligned} x &= \sqrt{y^2 - \frac{1}{16}} + \sqrt{z^2 - \frac{1}{16}} \\ y &= \sqrt{z^2 - \frac{1}{25}} + \sqrt{x^2 - \frac{1}{25}} \\ z &= \sqrt{x^2 - \frac{1}{36}} + \sqrt{y^2 - \frac{1}{36}} \end{aligned}$$

[2] This isn't technically an olympiad problem but still is an AIME #15!

and that $x+y+z=\frac{m}{\sqrt{n}}$ where m and n are positive integers and n is not divisible by the square of any prime, find $m+n$.
A-3. Prove that $BD-DC=\frac{c^2-b^2}{a}$.

2.1.2 Medians and the Centroid

A *median* AA' connects vertex A to the midpoint A' of side BC. Clearly, each median divides the area of the original triangle in half, since $[AA'C]=\frac{A'C\cdot h_a}{2}=\frac{A'B\cdot h_a}{2}=[AA'B]$. Just like the altitudes, the medians concur, at a point G known as the *centroid*.

When all three medians are drawn, six triangles are formed, GAB', GAC', GBA', GBC', GCA' and GCB'. An important result states that all six of these have the same area. The proof is simple: First, $[GAB']=[GCB']$ because GB' is a median of $\triangle AGC$. Similarly, $[GBC']=[GAC']$ and $[GBA']=[GCA']$. Now we can say that $[ABB']=\frac{1}{2}[ABC]=[ACC']$ and by subtracting $[AB'GC']$ we get opposite triangles $[GCB']=[GBC']$. Similarly, $[GBA']=[GAB']$ and by using these five equalities, we deduce that all six triangles have equal areas.

Unlike the altitudes, the medians have plenty of features that stand alone:
M-1. Prove that $AG=2GA'$.
M-2. Prove that if $\triangle ABC$ is on a Cartesian coordinate system, and $A=(a_1,a_2)$, $B=(b_1,b_2)$, and $C=(c_1,c_2)$, then $G=(\frac{a_1+b_1+c_1}{3},\frac{a_2+b_2+c_2}{3})$. In other words, G is the average of A, B and C.
M-3. Prove that the *medial triangle*, $\triangle A'B'C'$, is similar to $\triangle ABC$ and has an area of $[ABC]/4$.
M-4. Prove that G is the centroid of $\triangle A'B'C'$.
M-5. Prove that a triangle with two equal medians is isosceles.

2.1.3 Angle Bisectors and the Incenter

The final cevians we look at are the *angle bisectors*, or as some books call them, simply the bisectors. A bisector AA_1 is defined such that $\angle BAA_1=\angle A_1AC$ and A_1 lies on side BC. Like the altitudes and medians, the angle bisectors concur at a point known as the *incenter* I. Unlike them, we can easily prove this fact without waiting until Ceva's Theorem. We utilize an important fact about bisectors: Any point on the bisectors of lines l_1 and l_2 is equidistant from each line.[3] As a result, we know that the distances from any point on AA_1 to AB and AC are equal. The intersection I of AA_1 and BB_1 is thus equidistant from AB and AC and from AB and BC, so it is equidistant from AC and BC and lies on CC_1.

The fact that the distances from I to each of the sides are equal gives rise to the *incircle*, a circle centered at I with *inradius* r so that it is tangent to each of the sides.

Given a bisector AA_1, it is easy to see that angles AA_1B and AA_1C are supplementary, and thus have equal sines. By the Law of Sines, $\frac{AB}{BA_1}=\frac{\sin\angle AA_1B}{\sin\angle BAA_1}=\frac{\sin\angle AA_1C}{\sin\angle CAA_1}=\frac{AC}{CA_1}$. This result is known as the *Angle Bisector Theorem* and has applications in many non-olympiad contests such as AIME and ARML.
B-1. Prove that $[ABC]=rs$ where s is the semiperimeter, or $s=\frac{a+b+c}{2}$.
B-2. Prove the Angle Bisector Theorem in an alternative manner using the ratio $[AA_1B]:[AA_1C]$.

2.1.4 Excenters

Suppose that instead of using the internal bisectors of $\angle A$, we used the external bisectors (perpendicular to the internal ones). The three external bisectors would form a triangle $I_aI_bI_c$ with A on I_bI_c, etc. Moreover, in similar ways to above, we can prove that A, I, A_1 and I_a are collinear. I_a is

[3] The proof uses a reflection across each bisector which takes the diagram to itself. The converse is a bit harder.

equidistant from side BC and the *extensions* of sides AB and AC. (Prove as E-1!) Just as before with the incenter, we can draw *excircles* centered at each of the *excenters* I_a, I_b, I_c tangent to the sides or extensions. There is plenty to be said about the relationship between $\triangle I_aI_bI_c$ and $\triangle ABC$:
E-2. Prove that $\triangle ABC$ is the orthic triangle of $\triangle I_aI_bI_c$.
E-3. Prove that the tangent from B to the excircle opposite it has length s, the semiperimeter of $\triangle ABC$.

2.1.5 The Circumcenter

There is one last center that is not related to any cevians directly. That is the *circumcenter* O, the center of the *circumcircle*, the circle passing through A, B, and C. This circle has *circumradius* R. It is easy to see that O is the only point such that $OA = OB = OC = R$. The locus of points such that $OA = OB$ is the perpendicular bisector of AB, so O lies on that. The proof of the Law of Sines uses this circle and the properties of inscribed angles.

Now that we have introduced the four major centers, we can prove many things:
C-1. Prove that O is the orthocenter of the medial triangle $A'B'C'$.
C-2. Prove that G, H and O are collinear. This line is known as the *Euler line*. Moreover, prove that $GH = 2OG$.
C-3. Prove that the reflections of the orthocenter across each of the sides lie on the circumcircle.
C-4. Prove that $OI^2 = R^2 - 2Rr$.

2.1.6 The Nine-Point Circle

Many of the proofs in the previous part involve the homothecy from ABC to DEF, taking the centroid to itself and the circumcenter to the orthocenter. Let's consider a new homothecy, with scale of magnification 1/2 and center the orthocenter of the triangle. This takes the circumcircle to a new circle, which we call the *nine-point circle*. From the definition, the midpoints of AH, BH, CH lie on this circle. From C-3, the bases D, E, F of the altitudes also lie on it. In other words, it is the circumcircle of the orthic triangle. Finally, consider trapezoid $A'DC'B'$. Since $C'D = C'A = A'B'$ it is isosceles, and therefore (from CQ-3) cyclic. By applying the same argument to E and to F, the circumcircles of the orthic and medial triangles are the same. We have proven that nine points (yes, 9) associated with $\triangle ABC$ lie on one circle. These points are D, E, F, A', B', C', and the midpoints of AH, BH, CH. That is why it is called the nine-point circle. Its center N, as can be easily deduced above, is the midpoint of H and O, and also lies on the Euler line.
N-1. (Feuerbach's Theorem) Prove that the nine-point circle is tangent to the incircle and the three excircles.[4]
N-2. Prove that the radius of the nine point circle is equal to $\frac{1}{2}R$.

2.1.7 Stewart's Theorem

Having survived that half-semester of Euclidean geometry, we proceed to more fun stuff. Above, we applied the Law of Sines to the triangles formed by the bisector of $\angle A$; now, we consider the general cevian AX. Let $AX = d$, $BX = m$, and $CX = n$, so $m + n = a$. Using the Law of Cosines on sides AB and AC of $\triangle ABX$ and $\triangle ACX$ respectively, we get $\cos\angle AXB = \frac{d^2+m^2-c^2}{2md}$ and $\cos\angle AXC = \frac{d^2+n^2-b^2}{2nd}$. These angles are supplementary, so their cosines sum to zero: $0 = \frac{d^2+m^2-c^2}{2md} + \frac{d^2+n^2-b^2}{2nd} = \frac{d^2n+m^2n-c^2n+d^2m+n^2m-b^2m}{2mnd} \implies b^2m + c^2n = d^2(m+n) + mn(m+n)$, or the more memorable $bmb + cnc = dad + man$. We can apply these to the medians and angle

[4] In reality, the easiest way to prove this uses inversion, which we might do in February.

bisectors in general:
S-1. Prove that $m_a^2 = \frac{2b^2+2c^2-a^2}{4}$.
S-2. Prove that $l_a^2 = bc\frac{(b+c)^2-a^2}{(b+c)^2}$.

2.2 Ceva's Theorem

So how do we know the altitudes and medians concur? Ceva's Theorem is the simplest, most powerful method. It states that three cevians AX, BY, and CZ concur if and only if

$$\frac{BX}{CX}\frac{CY}{AY}\frac{AZ}{BZ} = 1.$$

The proof follows: Suppose these three cevians concur at P. Then $\frac{BX}{CX} = \frac{[ABX]}{[ACX]} = \frac{[PBX]}{[PCX]} = \frac{[ABP]}{[ACP]}$. Similarly, $\frac{CY}{AY} = \frac{[BCP]}{[BAP]}$ and $\frac{AZ}{BZ} = \frac{[CAP]}{[CBP]}$ and the result follows. To prove the other direction, simply suppose they do not concur and let P be the intersection of AX and BY. Draw in CP, meeting side AB at Z', then use the first direction to prove that $X = X'$.

The proofs that the medians concur is now trivial.
CT-1. Using Ceva's Theorem, reprove that the bisectors concur.
CT-2. Prove that the altitudes concur using Ceva's Theorem.
CT-3. The incircle is tangent to the sides of the triangle at A_2, B_2, and C_2, with A_2 opposite A, etc. Prove that AA_2, BB_2, and CC_2 concur. This point is known as the *Gergonne point*.

2.3 Pedal Triangles and Simson Lines

A *pedal triangle* XYZ is formed from the bases of the perpendiculars from a point P to the sides of a triangle ABC such that X lies on BC, etc. P can lie inside the triangle or outside it, or be on the sides. We have some interesting results from this setup:
PT-1. Prove that if $P = H$, then $\triangle XYZ = \triangle DEF$, and if $P = O$, then $\triangle XYZ = \triangle A_1B_1C_1$.
PT-2. Prove that $YZ = \frac{aAP}{2R}$, $ZX = \frac{bBP}{2R}$, and $XY = \frac{cCP}{2R}$.

The case where P lies on the circumcircle produces an interesting result. Without loss of generality, let $PABC$ be a cyclic quadrilateral in that order. From the right angles, $PYXC$, $PXBZ$, and $PZAY$ are cyclic just like $PABC$. With $\angle APC = 180^\circ - \angle ABC = \angle XPZ$ and by subtracting $\angle APX$, we get $\angle XPC = \angle ZPA = \angle XYC = \angle ZYA$. Therefore $\angle XYC = \angle ZYA$, so X, Y, Z are collinear; the pedal triangle is degenerate. The line XYZ is known as the *Simson line*.

We still can apply PT-2, but this time $XZ = XY + YZ$. Multiplying both sides by $2R$, we get $bBP = cCP + aAP$ or $AC \cdot BP = AB \cdot CP + BC \cdot AP$, which is Ptolemy's Theorem! Given the converse of Ptolemy's Theorem and the triangle inequality, Ptolemy's Inequality is easy to prove.

This is just the first of many things relating to Simson lines. Unfortunately, we simply don't have time to go as in-depth into them as *Geometry Revisited* does.

3 Circles

We're out of the triangle but won't be for long. Circles, besides being one of the simplest shapes, have some astounding properties not taught in most high school math classes.

3.1 Power of a Point

Power of a Point Theorem states that if two lines through P intersect a circle, the first at A and B and the second at C and D, then $PA \cdot PB = PC \cdot PD$. This value is known as the *power* of P

with respect to the circle. In the limiting case where $A = B$, it becomes PA^2.

Here's the proof: Consider tangent PA and secant PCD. $\angle ADC = \angle CAP$ since they subtend the same arc. From this we get $\triangle PAD \sim \triangle PCA$, so $\frac{PA}{PD} = \frac{PC}{PA}$, and the result follows. Now consider two secants PCD and PEF. From the previous proof, $PC \cdot PD = PA^2 = PE \cdot PF$, so that case is done. Finally, when you have two tangents, PA and PB, then $PA^2 = PC \cdot PD = PB^2$ so we are done.

PP-1. Prove that the power of a point distant d from the center of the circle is $d^2 - r^2$.

PP-2. Prove that if quadrilateral $ABCD$ is a circumscribed one, then $AB + CD = BC + DA$.

3.2 Radical Axis

The set of points with the same power with respect to two nonconcentric circles is known as those circles' *radical axis*. We will prove that the radical axis of two circles is a line perpendicular to the line connecting their centers (RA-1) in the lecture. If the circles intersect, the radical axis goes through the intersection points of the two circles.

If we have three circles, each pair has a radical axis. The intersection point of two of the axes has the same power with respect to all three circles, so it lies on the third radical axis, implying that they are concurrent at the *radical center* of those three circles.

This fails to work if the centers lie on a line. A special case exists when the two radical axes are actually the same, so the third is as well. Expanding beyond three, a set of circles, every pair of which has the same radical axis, is known as a *pencil of coaxal circles.*

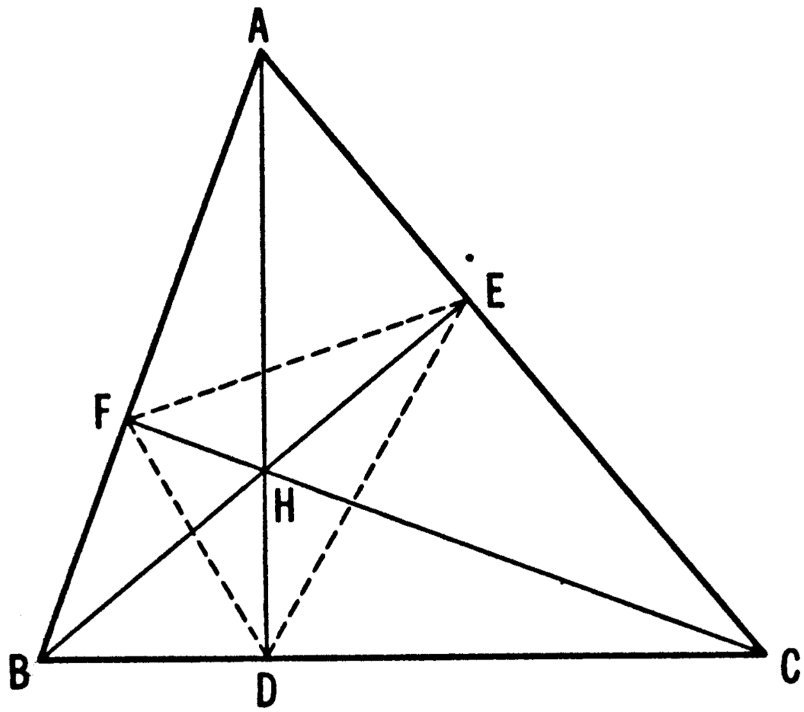
RA-2. Prove that the midpoints of the four common tangents to two nonintersecting circles lie on one line.

RA-3. Two nonconcentric circles are given. Prove that the set of centers of circles that intersect both these circles at a right angle is their radical axis (without their common chord if the given circles intersect).

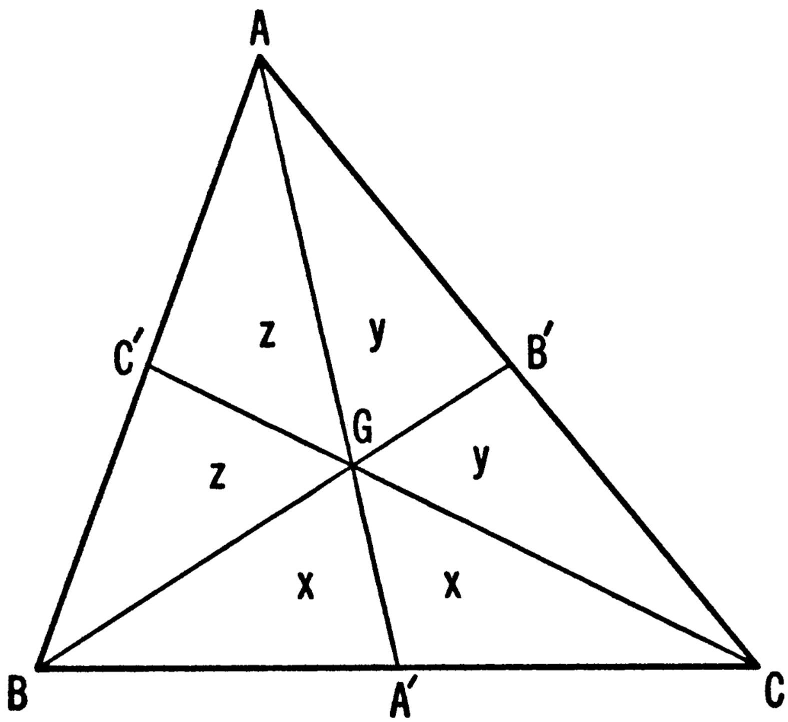
4 The next geometry lesson

With that, those are the fundamentals of olympiad geometry. We went a little more in-depth into triangles than we did into other topics, but this is not nearly all of geometry. There is so much more, such as the Menelaus' Theorem, spiral similarity, the Triangle Inequality, clines, barycentric coordinates, and straightedge constructions. These represent the several more areas we could go in depth on, but we have time for only one more, in February. At the end of today, we'll vote on what to do, from the following choices:

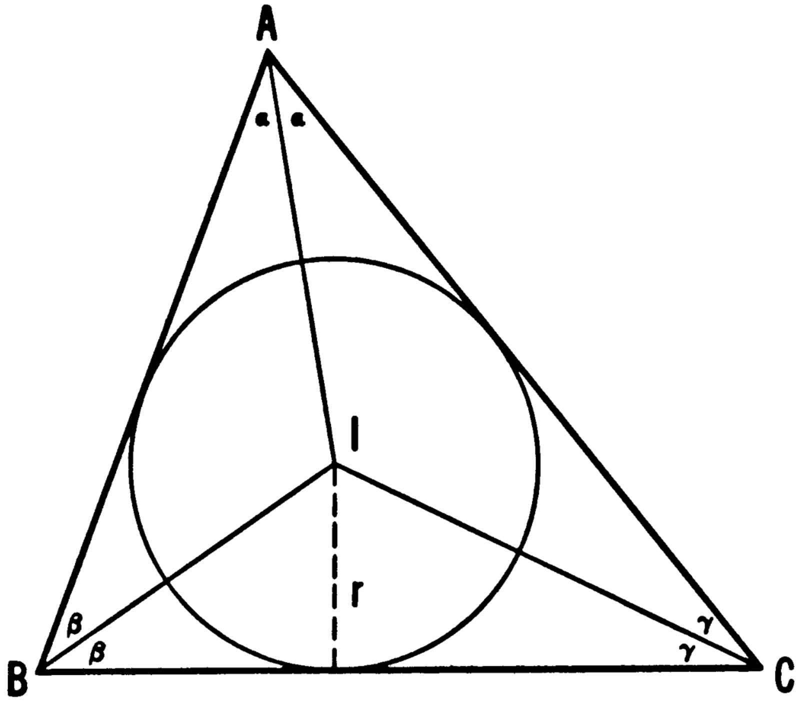
1. Geometric Inequalities (Ptolemy's Inequality is an example of these.)
2. Loci and Constructions
3. Coordinate Systems and Vectors
4. Collinearity, Concurrence, and Projective Geometry (like Ceva's Theorem)
5. Transformations
6. Inversion



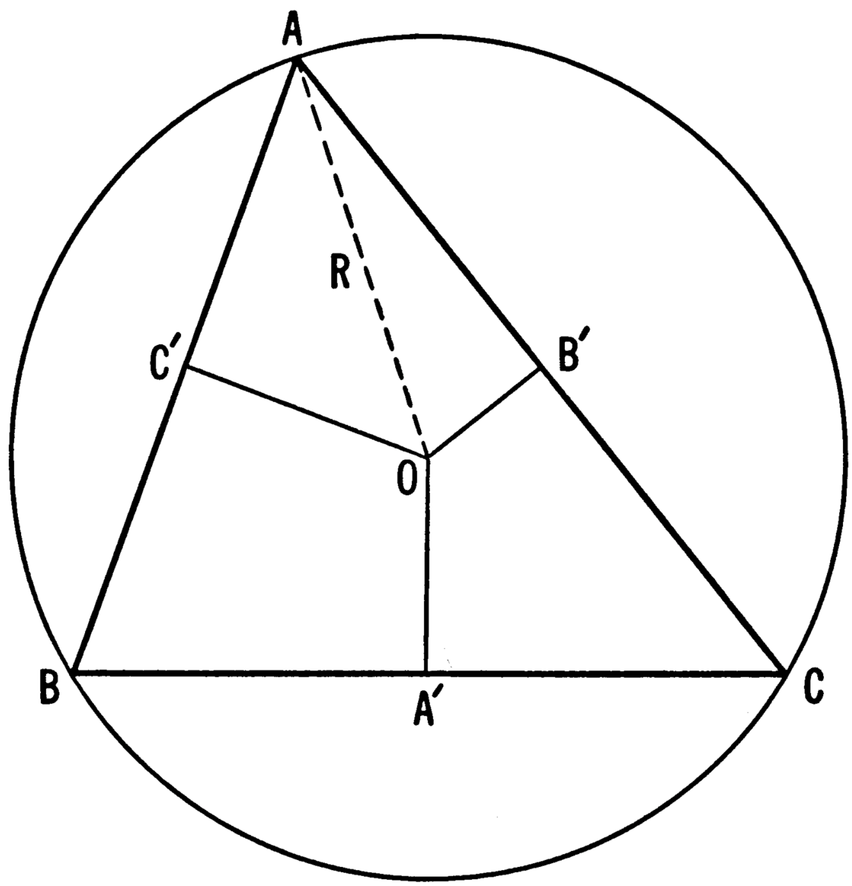
2.1.1 Altitudes and the Orthocenter



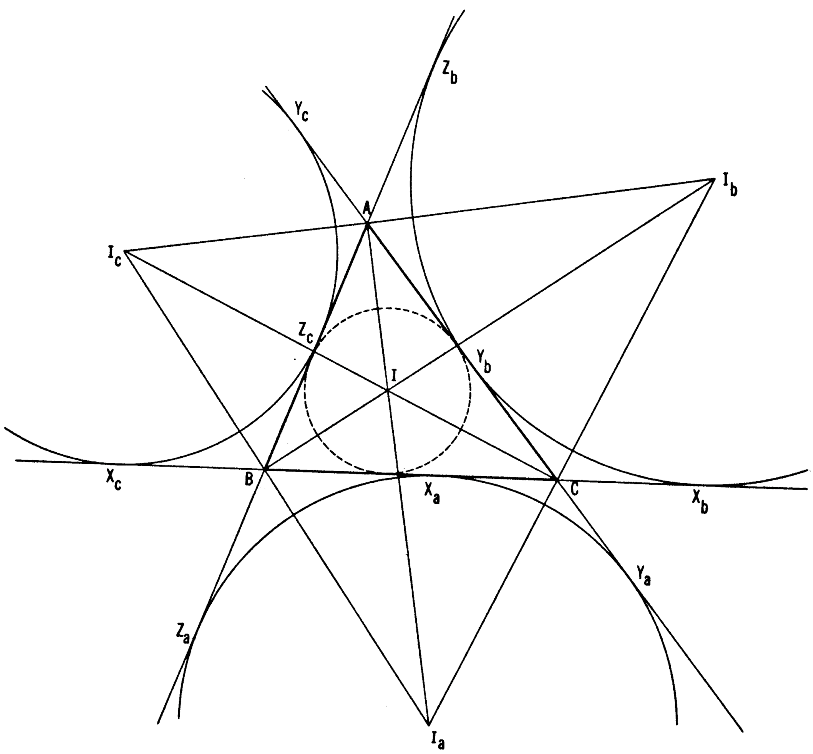
2.1.2 Medians and the Centroid



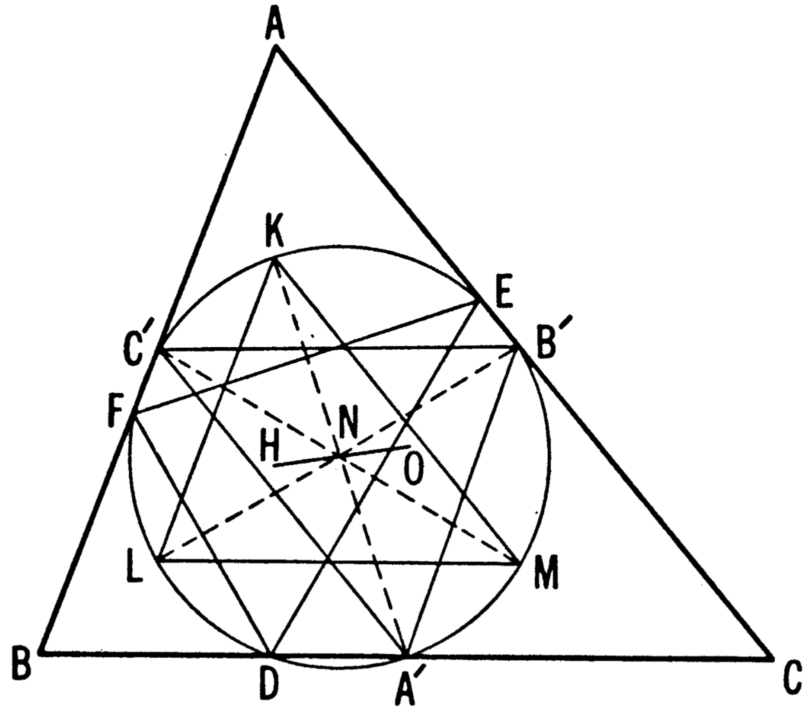
2.1.3 Angle Bisectors and the Incenter



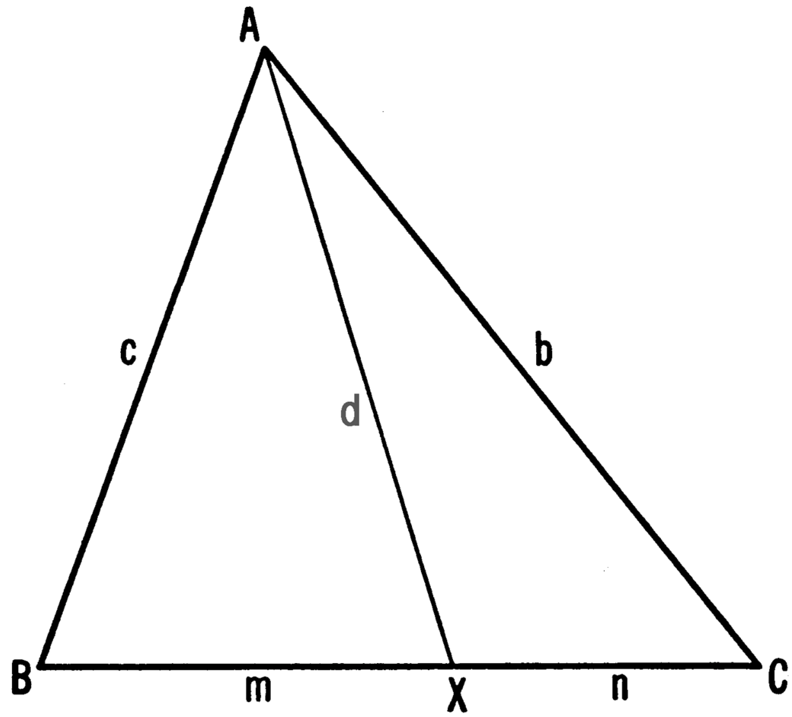
2.1.5 The Circumcenter



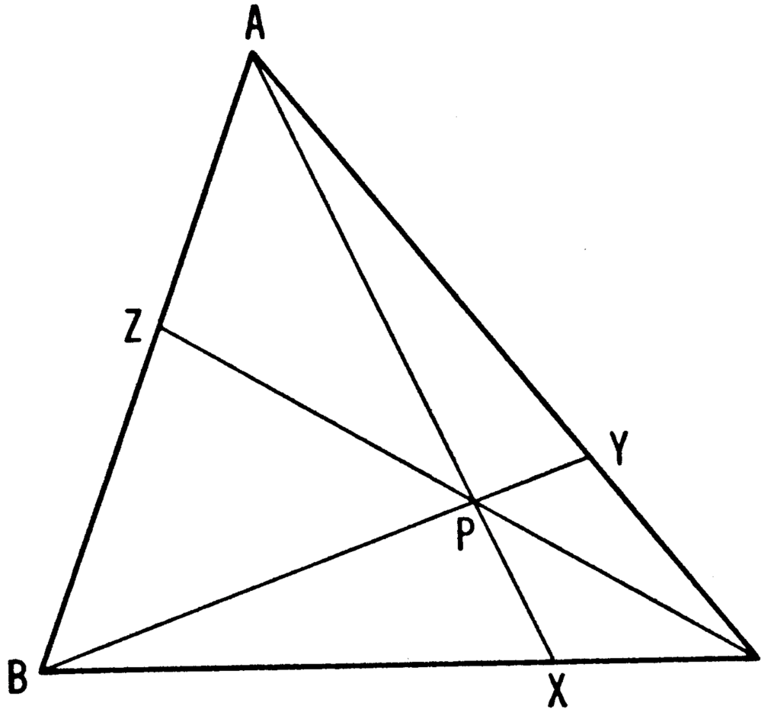
2.1.4 Excenters



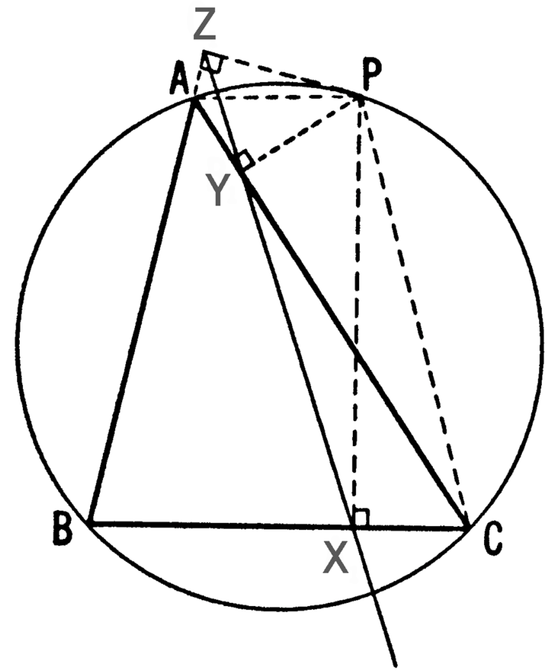
2.1.6 The Nine-Point Circle



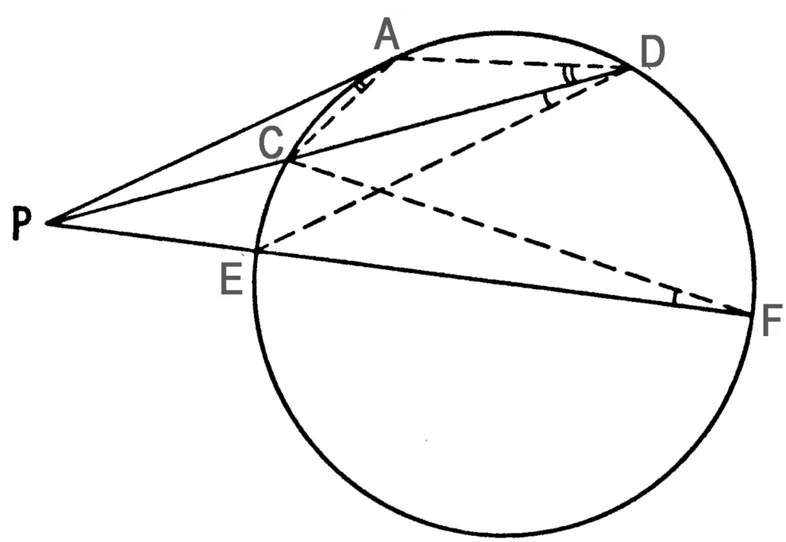
2.1.7 Stewart's Theorem



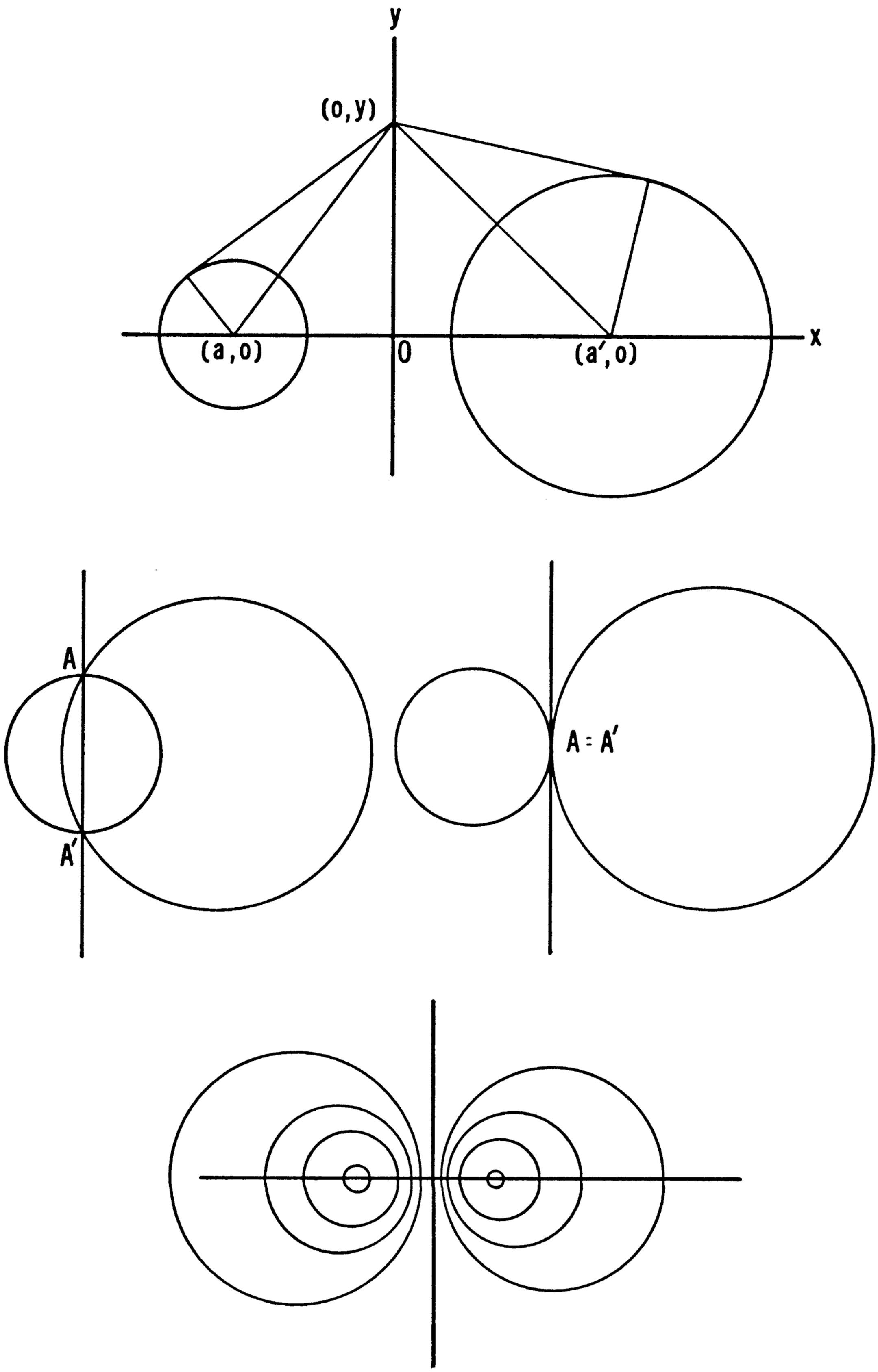
2.2 Ceva's Theorem



2.3 Pedal Triangles and Simson Lines



3.1 Power of a Point



3.2 Radical Axis