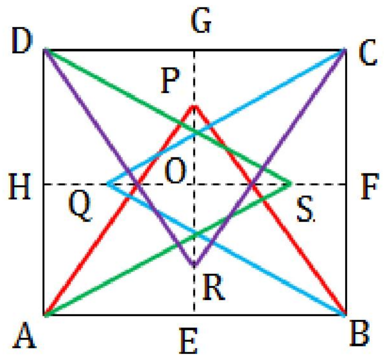
REGIONAL MATH'S OLYMPIAD (FIRST ROUND) 2011

SOLUTIONS

1. Let ABCD be a square of side 1 and P, Q, R, S be points inside the square such that APB, BQC, DSA, CRD are equilateral triangles. Compute the area of PQRS [12]

Sol:



Dropping PE ⊥ AB, QF ⊥ BC, RG ⊥ DC, SH ⊥ AD.

Let the point of their intersection be 'O'.

$PE^2 + AE^2 = AP^2$

$PE^2 = 1^2 - (0.5)^2$ [⊥ of an eq. Δ bisects the opp. side]

$PE = \sqrt{0.75} = 0.5\sqrt{3}$

Similarly $RG = 0.5\sqrt{3}$

Since G and E are mid-points of AB and CD,

GE || AD || BC.

PG + PE = AD

$PG + 0.5\sqrt{3} = 1$

$PG = 1 - 0.5\sqrt{3}$

Similarly $RE = 1 - 0.5\sqrt{3}$

$\therefore$ PG = RE

PR + PG + RE = BC

$PR + 2 - \sqrt{3} = 1$

$PR = \sqrt{3} - 1$

$QF^2 + FC^2 = QC^2$

$QF^2 + (0.5)^2 = 1^2$ [$\perp$ bisects the opp. sides]

$QF^2 = 0.75$

$QF = 0.5\sqrt{3}$

Similarly, $SH = 0.5\sqrt{3}$

$QH + QF = DC$

$QH + 0.5\sqrt{3} = 1$

$QH = 1 - 0.5\sqrt{3}$

Similarly $SF = 1 - 0.5\sqrt{3}$

$SF + QH + QS = 1$

$2 - \sqrt{3} + QS = 1$

$QS = \sqrt{3} - 1$

Since FH||DC,

$GO = 0.5 = OE = OH = OF$

$PO + GP = GO$

$PO + 1 - 0.5\sqrt{3} = 0.5$

$PO = 0.5\sqrt{3} - 0.5$

Similarly $PO = OQ = 0.5(\sqrt{3} - 1)$

$OP^2 + OQ^2 = PQ^2$

$0.5^2(\sqrt{3} - 1)^2 + 0.5^2(\sqrt{3} - 1)^2 = PQ^2$

$$2\left[\frac{0.25}{100}(4 - 2\sqrt{3})\right]$$

$2 - \sqrt{3} = PQ^2$

$PQ = \sqrt{2 - \sqrt{3}}$

Similarly,

$SR^2 = 2 - \sqrt{3}$

$SR = \sqrt{2 - \sqrt{3}}$

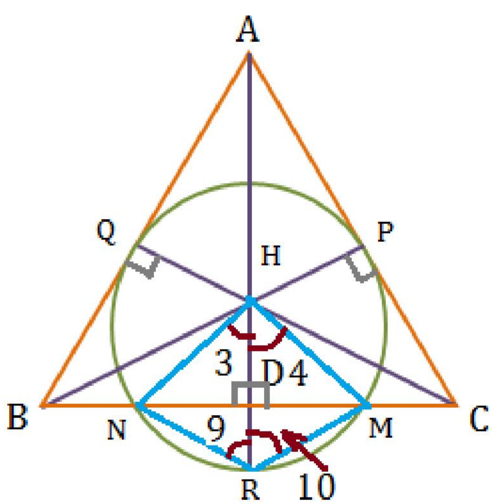
Area of square (Since opp. sides are equal, diagonals bisect at 90^0 and are equal)

$= \left[\sqrt{2-\sqrt{3}}\right]^2$

$= 2 - \sqrt{3}$

2. Let ABC be an acute angled triangle and D be the foot of perpendicular from A on BC. Let AD meet the ortho-circle of the triangle ABC in K and let H be the orthocenter of the triangle ABC. Prove that HD = DK. [10]

Sol:



In Δ HND and Δ HMD

HN = HM (Radius)

HD = HD (Common)

$\angle HDN = \angle HDM = 90^0$ (each)

Δ HND $\cong$ Δ HMD [By R.H.S congruency]

$\Rightarrow \angle 3 = \angle 4$ [By C.P.C.T.] ...(1)

ND = DM [By C.P.C.T.]

Now, In Δ NDK and Δ MDK

ND = MD [from above C.P.C.T]

$\angle$ NDK = $\angle$ MDK = 90^0 each

DK = DK common)

Δ NDK $\cong$ Δ MDK [By S.A.S. congruency]

$\angle 9 = \angle 10$ [By C.P.C.T.] ... (2)

NK = MK [By C.P.C.T.]

In Δ NHM and Δ NKM

NH = HM [Radius]

NK = MK [From C.P.C.T. (2)]

HK = HK [Common]

Δ NHM $\cong$ Δ NKM [By S.S.S. congruency]

$\therefore \angle 3 + \angle 4 = \angle 9 + \angle 10$ (By C.P.C.T.) ...(3)

Now

from (1) and (2) C.P.C.T.

$\angle 3 = \angle 4$

$\angle 9 = \angle 10$

$\therefore$ Equation (3) becomes

$\angle 3 + \angle 3 = \angle 9 + \angle 9$

$2 \angle 3 = 2 \angle 9$

$\Rightarrow \angle 3 = \angle 9$ .... (4)

In Δ NHD and Δ NKD

$\angle 3 = \angle 9$ [From 4]

$\angle$ NDH = $\angle$ NDK = 90^0 each

ND = ND [common]

$\Delta NHD \cong \Delta NKD$

$\therefore HD = DK$

Hence the result.

3. If a, b, x, y are four distinct real numbers such that $a^2 - b = b^2 - c = c^2 - d = d^2 - a$, prove that $(a + b)(b + c)(c + d)(d + a) = 1$ [10]

Sol:

$a^2 - b = b^2 - c$ [Given]

$a^2 - b^2 = b - c.$

$(a - b)(a + b) = b - c$

$(a + b) = \left[\frac{b - c}{a - b}\right]$ ....(1)

$b^2 - c = c^2 - d$ [Given]

$b^2 - c^2 = c - d$

$(b - c)(b + c) = c - d$

$b + c = \frac{c - d}{b - c}$ ...(2)

$c^2 - d = d^2 - a$ (Given)

$c^2 - d^2 = d - a$

$(c - d)(c + d) = (d - a)$

$(c + d) = \left[\frac{d - a}{c - a}\right]$ ...(3)

$d^2 - a = a^2 - b$ (Given)

$d^2 - a^2 = a - b$

$(d - a)(d + a) = a - b.$

$d + a = \frac{a - b}{d - a}$ ...(4)

On multiplying (1), (2), (3), (4)

$$(a+b)(b+c)(c+d)(d+a) = \left(\frac{b-c}{a-b}\right)\left(\frac{c-d}{b-c}\right)\left(\frac{d-a}{c-d}\right)\left(\frac{a-b}{d-a}\right)$$

$\therefore$ (a + b) (b + c) (c + d) (d + a) = 1.

Hence proved.

4. Find all pairs of integers (x, y) which satisfy the equation

$y^2(x^2+1)+x^2(y^2+16)=448.$ [15]

Sol:

$y^2x^2 + y^2 + x^2y^2 + 16x^2 = 448$

$2x^2y^2 + y^2 + 16x^2 = 448$

$2x^2y^2 + y^2 = 448 - 16x^2$

$y^2(2x^2 + 1) = 16(28 - x^2)$

If	$y^2 = 16$	then $2x^2 + 1 = 28 - x^2$
	$y^2 = 4\times4$	then $2x^2 + x^2 = 28 - 1$
	$y^2 = \sqrt{4^2}$	then $3x^2 = 27$
	$y = \sqrt{4^2}$	then $x^2 = 9$
	$y = \pm 4$	then $x = \pm 3$
If	$y^2 = 28 - x^2$	then $2x^2 + 1 = 16$
	$y^2 + x^2 = 28$	then $2x^2 = 16 - 1$
	$y^2 + x^2 = 28$	then $2x^2 = 15$
	$x^2 + y^2 = 28$	then $x^2 = 15/2$
	$x^2 + y^2 = 28$	then $x = \pm\sqrt{15/2}$

Since in this case, values of x are not integers, this, it is rejected

$\therefore x = \pm 3$ $y = \pm 4$

$\therefore$ possible pair = (3, 4); (3, −4); (−3, 4); (−3, −4).

5. Prove that for each positive integer m, the number 9.2^m can be written as a sum of three squares of positive integers. [12]

Sol:

9.2^m

For m = 1

$9.2 = 18 \rightarrow 4^2 + 1^2 + 1^2$.

By induction, we can assume for m = k

$9.2^k = x^2 + y^2 + z^2$.

Now, we have to prove for

m = k + 1

$2.9.2^k = x^2 + y^2 + z^2$

For m = 2

$9.2^2 = 36 \rightarrow 4^2 + 4^2 + 2^2$

By induction we can assume for m = k + 1

$9.2^{k+1} = x^2 + y^2 + z^2$

$2.9.2^k = x^2 + y^2 + z^2$

Hence Proved.

6. Determine how many distinct integers are there in the set

$$\left\{\left[\frac{1^2}{1998}\right],\left[\frac{2^2}{1998}\right],\left[\frac{3^2}{1998}\right],\ldots\ldots,\left[\frac{1998^2}{1998}\right]\right\}$$

Here [x] denotes the greatest integer less than or equal to x. [17]

Sol:

$$\left\{\left[\frac{1^2}{1998}\right],\left[\frac{2^2}{1998}\right],\left[\frac{3^2}{1998}\right]\ldots\ldots\ldots\ldots\left[\frac{1997^2}{1998}\right],\left[\frac{1998^2}{1998}\right]\right\}$$

Upto $\left[\frac{1^2}{1998}\right] = \left[\frac{2^2}{1998}\right] = \dots\dots\dots\dots = \left[\frac{44^2}{1998}\right] = 0$

Upto $\left[\frac{45^2}{1998}\right] = \left[\frac{46^2}{1998}\right] = \dots\dots\dots\dots = \left[\frac{63^2}{1998}\right] = 1$

$\therefore \left[\frac{64^2}{1998}\right] = 2$ and so on upto $\left[\frac{x^2}{1998}\right]$ where x = 999 and distinct integers are {0, 1, 2, 3, …………….. 499, } which are 500 in numbers

Now $\frac{(1000)^2}{1998} = \frac{1000000}{1998}$

$\frac{999^2}{1998} = \frac{998001}{1998}$ and

There difference is $(1000)^2 - (999)^2 > 1998$

so $\frac{(1000)^2 - (999)^2}{1998} > 1$

and onwards.

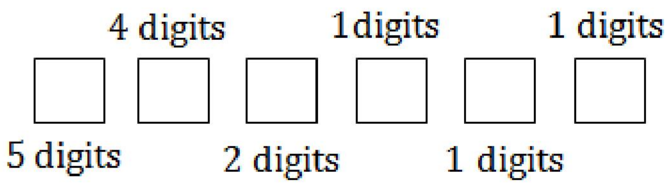
$\therefore$ x = 1000 and onwards every term will have a unique GINT, which can be 1998 – 999 = 999 numbers of unique integers. And the previous integers are 500 in number

$\therefore$ The total integers are 999 + 500 = 1499.

7. How many 6 digit numbers abcdef are there consisting of the digits 0, 1, 2, 3, 4, 5 each exactly once and satisfying the condition a + f = b + e = c + d? [12]

Sol:

0 + 5 = 1 + 4 = 2 + 3 = 5



The supplement of 0 = 5 and vice versa

The supplement of 1 = 4 and vice versa

The supplement of 2 = 3 and vice versa

The lakh's place can have any digit except 0. ∴ 5 possibilities.

Let's fix any one of the numbers at this place.

Thus, the one's digit can have only one digit i.e. its supplement

Now we are left with 6 – 2 = 4 digits.

The 10, 000's place can have any of the 4 digits. Lets fix any 1 number of the remaining at this place.

Thus, the 10's digit will have only one digit i.e. its supplement.

Now we are left with only 4 – 2 = 2 digits

the 1000's place can have any one of these two. Lets fix one of these there. Now we are left with only one digit.

∴ By rule of multiplication, we get,

Total numbers possible = 5 × 4 × 2 = 40 numbers.

8. How many ordered triples (a, b, c) of integers are there such that

(i) $1 \leq a \leq 10, 1 \leq b \leq 10, 1 \leq c \leq 10$ and

(ii) $a \leq b \leq c$? [12]

Sol:

$1 \leq a, b, c \leq 10$

$a \leq b \leq c$

when a = 1

$a = 1, c = 10$

$1 \leq b \leq 10$ $\boxed{10}$ cases

$a = 1, c = 9$

$1 \leq b \leq 9$ $\boxed{9}$ cases

$a = 1, c = 8$

$1 \leq b \leq 8$ $\boxed{8}$ cases

$a = 1, c = 7$

$1 \leq b \leq 7$ $\boxed{7}$ cases

$a = 1, c = 6$

$1 \leq b \leq 6$ $\boxed{6}$ cases

$a = 1, c = 5$

$1 \leq b \leq 5$ $\boxed{5}$ cases

$a = 1, c = 4$

$1 \leq b \leq 4$ $\boxed{4}$ cases

$a = 1, c = 3$

$1 \leq b \leq 3$ $\boxed{3}$ cases

$a = 1, c = 2$

$1 \leq b \leq 2$ $\boxed{2}$ cases

$a = 1, c = 1$

$1 \leq b \leq 1$ $\boxed{1}$ cases

Total cases = 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 55 cases.

Similarly	Total cases
For a =2	45
For a = 3	36
For a = 4	28
For a = 5	21

For a = 6	15
For a = 7	10
For a = 8	06
For a = 9	03
For a = 10	01
For a = 1	55
	220

∴ The total possible triplets are 220.