

60-odd YEARS of
MOSCOW MATHEMATICAL
OLYMPIADS

Edited by D. Leites

Compilation and solutions by G. Galperin and A. Tolpygo
with assistance of P. Grozman, A. Shapovalov and V. Prasolov

and with drawings by A. Fomenko

Translated from the Russian by D. Leites

Computer-drawn figures by

Abstract

Nowadays, in the time when the level of teaching universally decreases and “pure” science does not appeal as it used to, this book can attract new students to mathematics.

The book can be useful to all teachers and instructors heading optional courses and mathematical groups. It might interest university students or even scientists.

But it was primarily intended for high school students who like mathematics (even for those who, perhaps, are unaware of it yet) and to their teachers. The complete answers to all problems will facilitate the latter to coach the former.

The book also contains some history of Moscow Mathematical Olympiads and reflections on mathematical olympiads and mathematical education in the Soviet Union (the experience that might be of help to western teachers and students). A relation of some of the problems to “serious” mathematics is mentioned.

The book contains more than all the problems with complete solutions of Moscow Mathematical Olympiads starting from their beginning: some problems are solved under more general assumptions than planned during the Olympiad; there extensions are sometimes indicated. Besides, there are added about a hundred selected problems of mathematical circles (also with solutions) used for coaching before Olympiads.

The *Moscow Mathematical Olympiad* was less known outside Russia than the “*All-Union*” (i.e., National, the USSR), or the *International* Olympiad but the problems it offers are on the whole rather more difficult and, therefore, it was more prestigious to win at. In Russia, where sports and mathematics are taken seriously, more than 1,000,000 copies of an abridged version of a part of this book has been sold in one year.

This is the first book which contains *complete* solutions to all these problems (unless a hint is ample, in which case it is dutifully given).

The abridged Russian version of the book was compiled by **Gregory Galperin**, one of the authors of a great part of the problems offered at Moscow Mathematical Olympiads (an expert in *setting* olympiad-type problems) and **Alexei Tolpygo**, a former winner of the Moscow, National and International Olympiads (an expert in *solving* mathematical problems). For this complete English edition **Pavel Grozman** and **Alexander Shapovalov** (a first and a third prize winners at the 1973 and 1972 International Mathematical Olympiads, respectively) wrote about 200 new solutions each.

The book is illustrated by **Anatoly Fomenko**, Corresponding Member of the Russian Academy of Sciences, Professor of Mathematics of Moscow University. Fomenko is very well known for his drawings and paintings illustrating the wonders of math.

Figures are sketched under supervision of **Victor Prasolov**, Reader at the Independent University of Moscow. He is well-known as the author of several amazingly popular books on planimetry and solid geometry for high-school students.

From I.M. Yaglom’s “*Problems, Problems, Problems. History and Contemporaneity*”
(a review of MOSCOW MATHEMATICAL OLYMPIADS
compiled by G. Galperin and A. Tolpygo)

The oldest of the USSR Math Olympiads is the Leningrad High-school Olympiad launched in 1934 (the Moscow Math Olympiad runs since 1935). Still, for all these years the “most main” olympiad in the country was traditionally and actually the **Moscow Math Olympiad**. Visits of students from other towns started the expansion of the range of the Moscow Math Olympiad to the whole country, and, later, to the whole Earth: as International Olympiads.

More than half-a-century-long history of MMO is a good deal of the history of the Soviet high school, history of mathematical education and interactive work with students interested in mathematics. It is amazing to trace how the level of difficulty of the problems and even their nature changed with time: problems of the first Olympiads are of the “standard-schoolish” nature (cf. Problems 1.2.B.2, 2.2.1, 3.1.1 and 4.2.3) whereas even the plot of the problems of later olympiads is often a thriller with cops and robbers, wandering knights and dragons, apes and lions, alchemists and giants, lots of kids engaged in strange activities, with just few quadratics or standard problems with triangles.

Problems from the book compiled by Galperin and Tolpygo constitute a rare collection of the long work of a huge number of mathematicians of several generations; the creative potential of the (mainly anonymous) authors manifests itself in a live connection of many of the olympiads’ problems with current ideas of modern

Mathematics. The abundance of problems associated with games people play, various schemes described by a finite set, or an array of numbers, or a plot, with only qualitative features being of importance, mirrors certain general trends of the modern mathematics.

Several problems in this book have paradoxical answers which contradict the “natural” expectations, cf. Problems 13.1.9-10.2, 24.1.8.2, 32.7.3, 38.1.10.5, 44.7.3, and Problems 32.9.4 and 38.2.9.19 (make notice also of auxiliary queries in Hints!).

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Preface

I never liked Olympiads.

The reason is I am a bad sportsman: I hate to lose. Sorry to say, I realize that at any test there usually is someone who can pass the test better, be it a soccer match, an exam, or a competition for a promotion. Whatever the case, skill or actual knowledge of the subject in question often seem to be amazingly less important than self-assurance.

Another reason is that many of the winners in mathematical Olympiads that I know have, unfortunately, not been very successful as mathematicians when they grew up unless they continued to study like hell (which means that those who became good mathematicians were, perhaps, not very successful as human beings; however, those who did not work like hell were even less successful). Well, life is tough, but nevertheless it is sometimes very interesting to live and solve problems.

To business.

Regrettable as it is, an average student of an ordinary school and often, even the¹ teacher, has no idea that not all theorems have yet been discovered.

For better or worse, the shortest way for a kid to discover *mathematics as science*, not just a cook book for solving problems, is usually through an Olympiad: it is a spectacular event full of suspense, and a good place to advertise something really worth supporting like a math group or a specialized mathematical school. (Olympiads, like any sport, need sponsors. Science needs them much more but draws less.)

On the other hand, there are people who, though slow-witted at Olympiads, are good at solving problems that may take *years* to solve, and at inventing new theorems or even new *theories*.

One should also be aware of the fact that today's mathematical teaching all over the world is on the average at a very low level; the textbooks that students have to read and the problems they have to solve are very boring and remote from *reality*, whatever *that* might mean. As a Nobel prize winner and remarkable physicist Richard Feynman put it², most (school) textbooks are *universally lousy*.

This is why I tried to do my best to translate, edit and advertise this book — an exception from the pattern (for a list of a few very good books on elementary mathematics see Bibliography and paragraph H.5 of *Historical remarks*; regrettably, some of the most interesting books are in Russian).

If you like the illustrations in this book you might be interested in the whole collection of Acad.³ A. Fomenko's drawings (A. Fomenko, *Mathematical impressions*, AMS, Providence, 1991) and the mathematics (together with works of Dali, Breughel and Escher) that inspired Fomenko to draw them.

* * *

This is the first complete compilation of the problems from Moscow Mathematical Olympiads with solutions of ALL problems. It is based on previous Russian selections: [SCY], [Le] and [GT]. The first two of these books contain selected problems of Olympiads 1–15 and 1–27, respectively, with painstakingly elaborated solutions. The book [GT] strives to collect formulations of **all** (cf. *Historical remarks*) problems of Olympiads 1–49 and solutions or hints to most of them.

For whom is this book? The success of its Russian counterpart [Le], [GT] with their 1,000,000 copies sold should not deceive us: a good deal of the success is due to the fact that the prices of books, especially text-books, were incredibly low (< 0.005 of the lowest salary.) Our audience will probably be more limited.

¹We usually use a neutral “(s)he” to designate indiscriminately any homo, sapiens or otherwise, a Siamese twin of either sex, a bearer of any collection of X and Y chromosomes, etc. In one of the problems we used a “(s)he” speaking of a wise cockroach. *Hereafter editor's footnotes.*

²Feynman R. *Surely you're joking, Mr. Feynman*. Unwin Paperbacks, 1989.

³There were several scientific degrees one could get in the USSR: that of Candidate of Science is roughly equivalent to a Ph.D., that of Doctor of Science is about 10 times as scarce. Scarcer still were members of the USSR Academy of Sciences. Among mathematicians there were about 100 Corresponding Members — in what follows abbreviated to CMA — and about 20 Academicians; before the inflation of the 90's they were like gods. (This is why the soviet authors carefully indicate the scientists' ranks.)

However, we address it to ALL English-reading teachers of mathematics who could suggest the book to their students and libraries: we gave understandable solutions to ALL problems.

Do not ignore fine print, please. Though not as vital, perhaps, as contract clauses, *Remarks* and *Extensions*, i.e., generalizations of the problems, might be of no less interest than the main text.

Difficult problems are marked with an asterisk *.

Whatever the advertisements inviting people to participate in a Moscow Mathematical Olympiad say, some extra knowledge is essential and taken for granted. The compilers of [Le] and [SCY], not so pressed to save space, earmarked about half the volume to preparatory problems. We also provide sufficient Prerequisites. Most of the problems, nevertheless, do not require any special background.

The organizers of Olympiads had no time to polish formulations of problems. Sometimes the solutions they had in mind were wrong or trivial and the realization of the fact dawned at the last minute. It was the task of the “managers” (responsible for a certain grade) and the Vice Chairperson of the Organizing committee to be on the spot and clarify (sometimes considerably). Being unable to rescue the reader on the spot, I have had to alter some formulations, thus violating the Historical Truth in favour of clarity.

While editing, I tried to preserve the air of Moscow mathematical schools and circles of the period and, therefore, decided to season with historical reminiscences and clarifying footnotes. We also borrowed Acad. Kolmogorov’s foreword to [GT] with its specific pompous style. One might think that political allusions are out of place here. However, the stagnation and oppression in politics and social life in the USSR was a reason that pushed many bright (at least in math) minds to mathematics.

The story *A little problem*¹ and *Historical remarks* describe those times. Nowadays the majority of them live or work in America or Europe. I hope that it is possible to borrow some experience and understand the driving forces that attracted children to study math (or, more generally, to mathematical schools, from where many future physicists, biologists, etc., or just millionaires, also emerged). It was partly the *way* they studied and later taught, that enabled them to collect a good number of professorial positions in leading Universities all over the world (or buy with cash a flat on Oxford street, London).

What is wrong with the educational system in the USA or Europe, that American or European students cannot (with few exceptions) successively compete with their peers from the USSR? This question is so interesting and important that *The Notices of American Mathematical Society* devoted the whole issue (v. 40, n.2, 1993) to this topic, see also the collection of reminiscences in: S. Zdravkovska, P. Duren (eds.), *The golden years of Moscow mathematics*, AMS, Providence, 1993.

There were several features that distinguished mathematical circles and mathematical olympiads. The better ones were almost free of official bureaucratic supervision: all circles, olympiads, even regular lectures at mathematical schools (a lot of hours!) were organized by volunteers who often worked “the second shift” gratis for weeks or years (sic!); their only reward being moral satisfaction. There was freedom of dress code, possibility for children to address the leader of a circle, a Professor, by the first name (unheard of at regular schools), and the possibility for students who ran the circles and olympiads to ridicule the governing Rules in problems, without endangering the whole enterprise, by sticking the head out too far.

One of the problems (32.2.9.4 on “democratic elections”) was even published recently in a political magazin *Vek XX i mir (20-th Century and the World*, no. 10, 1991) with a discussion of its timeliness and realistic nature.

We should realize, however, that graduates of mathematical schools, though freer in thinking, were often handicapped by overestimation of professional (especially mathematical) skills of a person as opposed to humane qualities.

* * *

This compilation seemingly exhausts the topic: problems of the 70’s are often more difficult than interesting; owing to the general lack of resources Moscow Mathematical Olympiads became less popular. About 15 years ago similar lack of enthusiasm gripped famous Moscow mathematical schools. A way to revitalize mathematical education was suggested by one of the principal organizers of Moscow mathematical schools, Nikolaj Nikolaevich (Kolya) Konstantinov. It was similar to the most effective modern scientific way of getting rid of stafillacocus in maternity wards in our learned times: burn down the whole damned house. Konstantinov organized several totally new mathematical schools and a so-called *Tournament of Towns* (as a rival to counterbalance the Olympiads). The tournament became an international event several years ago; for the first collection see [T].

* * *

¹This story was published during an abortive thaw in 60’s; its author was unable to publish since.

I thank those who helped me: I. Bernstein, L. Makar-Limanova and Ch. Devchand; V. Pyasetsky, V. Prasolov and I. Shchepochkina. Pavel Grozman and Alexander Shapovalov had actually (re)written about 150 solutions each, Grozman made about a 1000 clarifying comments.

I also thank N. N. Konstantinov who introduced me to mathematics.

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Forewords

Mainly for the teacher. The problems collected in this book were originally designed for a competition, that is, to be solved in five hours time during an Olympiad. Many mathematicians in Russia were quite unhappy about this. They argued against this mixture of sport and science: many winners later did not achieve nearly so much in their studies as in this really very specific kind of “mathematical sport”. Vice versa, many people who could never succeed under stress proved later to be among the most talented and productive. It is true also that real mathematics deals mostly with problems taking months and years, not hours, to make a step forward.

Still, for many schoolchildren, the idea of a competition is very attractive, and they can take part just for its sake and so discover how diverse and interesting Mathematics (not just math) can be. Afterwards one can find a lot of more productive mathematical activities than competitions: reading mathematical books is just one. But there should be the very first step, and Olympiads, as well as Olympiad style problems in school mathematical clubs and such, help to make it.

One can use this book as the source of problems to organize an Olympiad-like competition on one’s own, or for the group or individual studies. In Moscow the same group of the University¹ professors and postgraduate students that launched the Olympiads (see *Historical Remarks*) also established a tradition of “mathematical circles” — weekly gatherings of schoolchildren at the University, where they can attend a lecture, solve some problems, report their progress and get advice. Many of the problems first proposed at the Olympiad later became the “circles’ folklore” and taught several generations.

To use these problems in this way is probably much better, because it is up to a student to choose: either to compete with others for the number of problems solved, or just to besiege a single difficult one. Thus, different psychological types can be properly treated without hurting anybody. (A failure at the Olympiad can be a cause for a grave psychological disturbance in the whole future life.)

Some problems are tremendously difficult²; only few individuals could solve such problems. As you may learn from *Historical Remarks*, there were several problems with not a single correct solution presented to the Organizing Committee (while the Committee only knew a wrong solution). Therefore, never mind if you try to crack some of these hard nuts and fail: so did many others. Try it again later or look up Solutions: perhaps you just misunderstood the formulation. Just do not try a new problem on your pupils before examining it yourself properly: it may save a teacher a lot of trouble.

You may encounter some difficulties trying to explain solutions to your pupils due to the curriculum differences in the U.S. and S.U. You can find feeble consolation in the fact that your colleagues in Russia experience the same difficulties: three more or less radical reforms have passed since the first Olympiad, and the fourth catastrophe is in progress. However, the authors tried to use wherever possible only “elementary” mathematics in solutions, though throwing in a little Calculus could have made it much easier.

We hope that the spirit of the Moscow Mathematical Olympiads will remain the same and that for many years to come there will be ringing voices of teenagers in the rooms of Moscow University and questions will be asked again: “When will the next Olympiad be held?”³

¹M. V. Lomonosov Moscow University is, or rather was before the mass emigration of the '80s, for the USSR more than what Princeton and Harvard combined are for the USA, at least as regards mathematics. Mathematics was also well taught in some of Moscow Institutes but the study there was handicapped by the red tape and the general lack of the “air”. At the moment the major part of Institutes in Moscow and larger cities are renamed into “universities”, but still The University remains outstanding.

²Sometimes so much so that even after 9 years of editing and re-editing, nobody knew the answers; to a couple of problems we only knew a wrong answer. All this, together with the correct answers became clear when Pavel Grozman, a First prize winner at the 1973 International Mathematical Olympiad lent a hand. Several mistakes (with corrections) were discovered by A. Shapovalov, V. Prasolov and V. Pyasetsky.

³Or, rather, more usual “Will we be allowed to eat during the Olympiad?”

Mainly for students. This book may be useful for you in your studies and it may be an entertainment.

It may sound curious for those who know only usual text-books but a lot of students of your age get a lot of fun just solving mathematical problems. To feel this joy for the first time, one usually has to taste something very different from the common kind of school algebra and geometry.

The authors of the problems collected here tried their best not to be boring or scholastic; they preferred rather to be mocking and ridiculing.

There is a lot of good sense in these problems, too. School mathematics is usually formulated in a very specific “scientific” (pseudo-scientific far too often) way. You can recognize a school manual phrase in a hundred. But, in real life, nobody will prepare your problems for you in such a manner¹: you will have to distill from an actual, vaguely put, problem a precise mathematical one yourself. So the stranger-looking problems teach you to recognize mathematics in the world around you.

Finally, while solving these problems you can get acquainted with many ideas and notions, quite common for mathematics of this century but still not popular enough with school curricula. Without bothering about strict terms, you will learn how to deal with many principles of the so-called “discrete” mathematics, which proved to be a universal language for all natural sciences.

The syllabus of mathematical studies at Soviet secondary schools has undergone in the course of history several radical changes. For instance, the translator of this book was taught complex numbers and trigonometry but not integrals whereas the next and previous generations enjoyed the opposite choice. Some of these changes were akin to smashing blows².

We have tried to solve the problems using elementary school mathematics; some of the solutions would, however, be too long if presented in elementary terms so we used some calculus. Ask your teacher in case of confusion and do not blame him/her if (s)he fails to solve some of the problems. In the awful case that a fault or misprint crept into the text please send a tip to the editor or compilers.

One of the points we’d like to make is that ability to solve Olympiad problems does not distinguish a good teacher nor a good mathematician; speaking mathematically this is neither a necessary nor sufficient condition. A good adult mathematician, however, can usually solve any Olympiad problem³, at least by more advanced means.

You must know that some of the problems collected here are very complicated. Some even proved to be so difficult that in 5 hours of the Olympiad none of the students in the ten-million-city of Moscow succeeded in finding the correct solution. Such a problem can astound even Ph. D. holders⁴. So you should not consider yourself (or your schoolmate, or your professor) inadequate if you (or they) do not make progress even after a week-long struggle.

But you should not be scared off! Our advice: set a difficult problem aside for a while rather than rush immediately to *Solutions* after an unsuccessful first attack.

Now you can begin without further delay. Be sure to skim through *Prerequisites* first!

More advice: always put down your solutions to stew for a while. Discuss them, if possible, with your teacher and classmates. Afterwards, if you have found no faults in your proof, read the one proposed in this book. It may well differ from the one you invented yourself (and if it is similar to yours ... well, the greater chance for both to be correct).

Remember that an answer, such as “Yes”, “Never” or “Five hundred and five”, is not a solution, even if correct. *Proofs*⁵ of all your statements are expected, and the word *clearly* should not litter it: try to explain everything. Be aware of the fact that when *we* wrote “clearly, ...”, “it is easy to see”, “it is not difficult to verify”, etc., *we meant* that it is the inquisitive reader who will actually complete the proof. (Otherwise the book would have been twice its size and price.)

Use Part 2 (*Solutions*) to see if something you thought to be obvious can indeed be deduced from some much more evident facts.

* * *

¹This is a typical stand of an ‘olympiadchik’, but it is not altogether wrong.

²This might give to an optimistic student an idea that either of the above choices should be harmlessly sacrificed in favor of some other, seemingly healthier than math, activity.

³If has nothing better to do; a mathematician usually solves Olympiad-type problems around the clock, anyway; this is one’s job and hobby.

⁴Some of the problems have been correctly solved for the first time here.

⁵An average student does NOT know what exactly this word means; neither do some (too many) school teachers. May this book help you to understand this.

Unlike exercise from Problem books, a real problem requires to be investigated: one has to find out at least a way to tackle it. Therefore, start with easier problems. Do not solve all the problems in a row; this is not a homework, choose the problems more interesting to you.

If you can not solve the problem, try to make a similar but easier problem and solve *it*. If you can not even that, read a hint. If it does not help either, try not to gulp the solution but read it slowly, as a detective story in which you try to guess the next turn of mind. Finally, look at the solution “in the large”: what are its main driving ideas, and, most important, how could one get to it.

If you managed to solve the problem, read its solution we offer, since it is instructive to compare different solutions (even if one of them is wrong).

To understand a solution deeper, ask yourself: at what stages of the proof we used different given data? Will the statement be true if we slacken or omit a condition? Is the converse statement true?

Important is not the quantity of problems solved but the deepness of understanding their solution, the new information acquired.

AT AN OLYMPYAD

1. Read all the problems offered and order them as you will solve them. Bare in mind that the order given is usually in accordance with their difficulty from the compilers’ point of view.

2. If the problem has a too easy solution, then, most probably, you misunderstood the formulation or made a mistake.

3. If you can not solve the problem, try to simplify it: make smaller numbers, consider particular cases, etc., or solve it “backwards”, by the rule of contraries, substitute indeterminates instead of given numbers or the other way round, etc.

4. If undecided whether a statement is true, try alternately to prove and to disprove it.

5. Do not stick to one problem for too long: from time to time make a break and estimate your position. If you managed to advance, continue, otherwise, leave the problem for a while.

6. If tired, relax immediately (look at the sky and contemplate the infinite or walk along the corridor).

7. Having solved the problem, immediately write it down in proper official style, not as a letter to a pal. This will help to verify the arguments and will free the old bean for other problems.

8. Each turn of idea should be documented even if it looks obvious. It is convenient, therefore, to express the solution as a series of statements (lemmas).

9. The student seldom rereads his/her own production *trying to put oneself into the jurors’ shoes: will anybody be able to understand anything you wrote?”*

Good luck and best ideas!

Acknowledgments. We deem it our pleasant duty to point out about 40 years of Sisyphus’ work on mathematical education performed by N. N. Konstantinov.

Konstantinov was (and still is) one of the principal organizers of the specialized Moscow’s mathematical schools, instrumental in arranging Moscow Olympiads and other mathematical contests (Tournament of Towns, etc.). He always was their soul.

Acad. A. N. Kolmogorov, who always actively participated in organizing Moscow Mathematical Olympiads from their start till his death, did much for the book [GT] as its editor and scientific consultant. We use this opportunity to express our warmest gratitude to him.

We are also obliged to all those who helped us in working on the book and preparing it for publication, and above all to V. V. Prasolov, V. M. Tikhomirov, N. B. Vasiliev, and A. M. Abramov, as well as A. P. Savin, S. M. Saakyan, A. L. Toom, E. A. Morozova, R. S. Cherkasov, and A. B. Khodulev.

We are grateful to V. G. Boltiansky and I. M. Yaglom, and to A. A. Leman for their kind permission to use parts of their article about the book [YB] and borrow from [Le], respectively.

Academician A. N. Kolmogorov's foreword to [GT]

Our country needs many research mathematicians who are able to make discoveries in mathematics itself and to apply it in unusual ways that require great ingenuity. Usually, scientists who started to practice research-type activity while still at school were more successful later on. Many of them made serious discoveries when 17–19 years old. To postpone the involvement of young people in intense research is to irrevocably lose many of potentially very creative researchers.

Addressing school students who are seriously thinking of becoming real mathematicians, I will tell them the following. Just as in sports, practice requires plenty of a young mathematician's time. It will be profitable if you peruse this collection of problems on your own, choose a problem whose formulation seems interesting to you and start thinking it over without reading the solution.

Do not be afraid that you may waste many, many hours doing that. In this respect I recall the words of Boris Nikolaevich Deloné, one of the most remarkable Russian mathematician, who said that a great scientific discovery in mathematics differs from a tough Olympiad problem only in that the problem takes 5 hours to solve while an important research consumes 5 000 hours. Deloné liked to exaggerate; do not take these "5 000 hours" too literally. But it is typical of a mathematician who attacks a difficult problem to be able to ponder over it for days. If a problem proves a hard nut to crack it is reasonable to try another one. But it is also good to turn back to the first one after a while. It is sometimes useful even for mature mathematicians to put off a difficult problem for some time. It often happens that a solution suddenly emerges from the subconscious after a period of time.

It is only natural that one is delighted and even proud of his/her success at an Olympiad. But failure should not upset you too much or make you disappointed in your abilities in mathematics. The success at an Olympiad requires certain special talents which are not at all necessary for a successful research. The very fact of strict limitation of time allotted for solving problems during an Olympiad makes many people quite helpless. There are, however, mathematical problems whose solution can only be obtained as a result of a very long and calm contemplation and after moulding new concepts. Many problems of this kind were solved by Pavel Sergeevich Alexandrov who used to say that if there were mathematical olympiads in his time he may have never become a mathematician since his main accomplishments in mathematics resulted from a long and deep contemplation rather than a fast-working smartness.

I hope that our collection of problems will be of great help for all instructors of math clubs and for the organizers of local olympiads. I wish to make two comments for them.

The Moscow Mathematical Olympiads were originally addressed to 9–10 graders. Since 1940, however, 7-th and 8-th graders were also invited. I think this choice of age group is quite justified. It is at this age that the knack for mathematics becomes manifest. Certainly, one can organize olympiads for younger kids but one has to bear in mind that most of the boys and girls who distinguished themselves in problem-solving contests in 5–6 grades lose their special capabilities and even interest in mathematics as they grow up¹.

When organizing an olympiad for a particular group of students, it is very important to correctly estimate in advance the complexity of the problems to be offered. These should be planned so that the most capable participants could solve most of the problems but there should not be too many participants who failed to solve at least one problem. Some information about the problems which, unexpectedly, proved to be too difficult in practice can be found in reports on Olympiads published in the magazines *Matematika v shkole* and *Kvant*². Regrettably, the level of difficulty was not always correct at some of Moscow Mathematical Olympiads. The content of the problem usually was, nevertheless, up to very high standards.

In *Historical remarks* the authors describe in detail the great experience of Moscow Mathematical Olympiads and how the process of devising olympiad-type problems went hand-in-hand with the work of mathematical clubs under the Moscow University's egid. The joint efforts of the leaders of the University's math circles in a great and outstanding job. It resulted in a book you are going to read now.

The job of the compilers, G. A. Galperin and A. C. Tolpygo, is wonderful and deserves deep gratitude.

¹This argument seems doubtful; more serious troubles are (a) the strain and stress of an Olympiad which is the real danger for students at the early age and (b) the difficulty for the organizers to devise reasonably tough and more or less meaningful problems at the level needed.

²It is a very good mathematical magazin and during its first 20 years it was a REMARKABLY GOOD magazin. Now a very close version to the Russian original is published in English as *Quantum*.

Part 1: Problems

Introduction

Prerequisites and notational conventions

The following prerequisites were largely assumed to be known to any participant of an Olympiad.¹ Lately it became clear that the gap between the standard school mathematics and that of an Olympiad should be bridged in order not to discriminate against an average student. For example, the collection² of preparatory problems for the jubilee 57-th Olympiad contained several very useful comments partly coinciding with ours. We borrowed some of them.

We expect that the reader of knows how to plot the graph of the function $y = ax^2 + bx + c$ given the coefficients a, b, c .

Various (good) books on elementary mathematics written in English use different notations, e.g., quadrangle — quadrilateral; cathetus — leg, etc. To augment the confusion the original problems for various Olympiads were compiled by different authors, each with the own style.

We edited the text in order to reduce such discrepancies but to please all was impossible. For example, the requirements of present AMS mathematics editors to style are sometimes at variance with Webster's dictionaries and differ from guidelines formerly advocated by AMS via Halmos's pamphlet "How to write mathematics", originally published in L'Enseignements Mathématiques t.XVI, fasc. 2, 123–152 and reprinted many times since then in many languages.

Problems are enumerated as follows: the *first* number is *the number of the Olympiad*, the *second* one is *the number of the tour* (if there was only one tour this number is skipped), the *third* number is that of *the grade*, and the *fourth* number is *the number of the problem itself*. There are natural modifications of these notations, e.g. **1.2.C.1** denotes Olympiad 1, tour 2, set C, Problem 1; **4.2.2** denotes Olympiad 4, tour 2, Problem 2; **10.2.7-8.3** denotes Olympiad 10, tour 2, grades 7-8, Problem 3; in **33.D.7.4** D is for Pythagoras' Day.

An asterisk marks a more difficult (heading of a) problem, e.g., 1.2.C.1 b)*.

The principles. Dirichlet's principle. *If n rabbits sit in k hutches, then there is a hutch with not less than $\frac{n}{k}$ rabbits and a hutch with not more than $\frac{n}{k}$ rabbits.*

Though this principle is obvious, it sometimes solves difficult problems: it is not always easy to select objects that play the role of rabbits and hutches.

The Dirichlet's principle applies to continuous quantities as well: *If n rabbits have eaten k kg of food, then there is a rabbit who has eaten not less than $\frac{n}{k}$ kg and a rabbit who has eaten not more than $\frac{n}{k}$ kg.*

The principle of mathematical induction is used to prove an infinite sequence of statements:

Consider a statement $S(n)$ that depends on a positive integer $n \geq n_0$. We believe $S(n)$ to be true for any positive integer $n \geq n_0$ if

- 1) $S(n_0)$ holds for some n_0 ;
- 2) the validity of $S(l)$ for $n_0 \leq l \leq k$ implies $S(k+1)$.

Heading 1) is called the *base of induction*; heading 2) is called the *inductive step* and the assumption we use in 2) *the inductive hypothesis*³.

Example: *Find the sum $1 + 3 + \dots + (2n - 1)$.*

Solution. Let us denote this sum by $S(n)$ and look at it for small n . We see that $S(1) = 1$, $S(2) = 4$, and $S(3) = 9$. An educated guess is: $S(n) = n^2$.

The base of induction is fulfilled for $n_0 = 1$.

Now the inductive step: $S(k+1) = S(k) + 2(k+1) - 1 =$ (by the inductive hypothesis) $= k^2 + 2k + 1 = (k+1)^2$. Q.E.D.

Sometimes the induction is used *backwards*, cf. Problem 20.2.10.5. Namely, *Consider a statement $S(n)$ that depends on a positive integer $n \geq n_0$. We believe $S(n)$ to be true for any positive integer $n \geq n_0$ if*

¹Although every cultured person must know these facts, among other useless data, some people do not; therefore, we recommend to skim through this section before advancing further.

²Kanel-Belov A., Kovaldzhii A., Vasilev N. *Preparatory Problems to LVII Moscow Mathematical Olympiad*, Treade, Moscow, 1994.

³Some mathematicians with quite unorthodox minds doubt the universality of this principle. They only believe in the numbers we can *actually count*. Since the mathematics obtained this way is supposed to be very poor, these ideas are not popular. They did not die out, however, because the attempts to consider only *constructible* statements clarified some messy or nonconstructive proofs of interesting theorems.

- 1) $S(n_0)$ holds for some n_0 ;
- 2) the validity of $S(l)$ for $k \geq l$ implies $S(k - 1)$.

Warning. It could happen that the inductive step is easy to perform but the conclusion is nevertheless wrong. This happens if the justification of the base of the induction is ignored and we are trying to prove a wrong statement.

Example: Let us “prove” that *The eyes of all people are of the same color*. Indeed, the eyes of one person are of the same color¹. Now, assume that the statement is true for k people; then it is obviously true for $k + 1$ people since any k of these $k + 1$ persons have the eyes of the same color. The catch is that for $k = 2$ the statement is generally false.

Games: selected ideas. 1) *Solution backwards*, cf. Problems ...

2) *Correspondence*. The presence of a lucky move can be justified by a symmetry, partition into pairs or a complement, cf. Problems ...

3) *Transfer of the move*. If we can use the opponent’s strategy we are not worse off than the opponent. For example we win or draw if we can assume the opponent’s winning position at will, cf. Problems...

4) “*Prepared homework*” (on an infinite field), cf. Problems

Selected ideas. *Reductio ad absurdum*. If we assume that the statement to prove is false and deduce a contradiction from this assumption, this will prove that the statement was true after all.

Estimates. We estimate a complicated quantity with a simpler one. The inequality between the mean arithmetic and the mean geometric is often used.

An *invariant*. A quantity is sometimes always even (odd) or just a constant. This implies that a situation in which this quantity is odd (even) or not a constant are impossible, cf. Problems ... Sometimes a quantity can be calculated (or estimated) in two ways and we compare the results, cf. Problems A residue can serve as an invariant and we only have to check the possibilities case-by-case, cf. Problems ...

Cycles or *periods* that arise in a *process* are examples of invariants, cf. Problems ...

The rule of an extreme element. Singular or *extreme* objects (the largest number, the nearest point, the vertex of a polygon, the degenerate circle, the limit case, etc.) often clarify the regular case. Cf. Problems ...

Standard common notations. In all problems on tournaments it is assumed that each participant competes with every other only once. In a chess tournament, a player gets 1 point for victory, half a point if the game ends in a draw, and 0 for loss; in soccer, all scores are twice as much. In basketball, tennis, etc., there are no draws.

The *main diagonal* (of a square array, or a table, or a matrix) is the one which connects the top left corner with the bottom right corner while the other longest diagonal is called the *side diagonal*. Dimensions $m \times n$ of a table show that it has m rows and n columns.

In all problems on *graph* or *checkered* paper or plane, we assume that all small squares or cells are uniform squares of side 1, and any *vertex* or *node* is the intersection of any two non-parallel lines of the grid, i.e., is a vertex of a square.

A *tableau* or just *table* is a rectangular piece of graph paper cut along the lines of the grid.

Space means \mathbb{R}^3 , i.e., our usual (mathematically speaking, Euclidean) three-dimensional space in which for any two points the *distance* between them is defined.

In all problems involving light rays, billiard balls, etc., we assume that *the angle of reflection is equal to that of incidence*.

\overline{abc} denotes the positive integer whose (decimal, usually) digits are a, b, \dots, c .

An expression of the form $\underbrace{aa\dots a}_{1993}\underbrace{bb\dots b}_{3991}$ means that a is repeated 1993 times and b is repeated 3991 times. Sometimes we write this explicitly if space permits.

\emptyset denotes the *empty set*, i.e., , the set without elements; we assume that \emptyset is a subset of any set.

$M \subset N$ denotes that every element of the set M belongs to N ; we say that N is a *subset* of M ; $s \in S$ denotes that the *element* s belongs to the set S ; $A \setminus B = \{a \in A : a \notin B\}$ denotes the *set-theoretic difference* of sets A and B .

The *intersection* of the sets M and N is denoted by $M \cap N = \{x : x \in M \text{ and } x \in N\}$; the *union* of two sets M and N is denoted by $M \cup N = \{x : x \in M \text{ or (not exclusive) } x \in N\}$; a *disjoint union* is the union of nonintersecting sets; we often consider intersections and unions of several sets.

¹Even this is sometimes wrong, but for the sake of argument we will consider such deviations as aliens, not humans.

A *partition* of a set is its representation as the disjoint union of its subsets. An example: *coloring* each element of the set in one color.

A (finite or infinite) family of sets A_1, A_2, \dots is a *covering* of a set M if every point of M belongs to some A_i . One point of M can be *covered* several times (by different A_i .) A *tiling* is a covering (usually with identical sets) such that each point of M is covered exactly once.

Often (but not in this book) the description of a set $\{a_i\}_{i \in I}$ whose elements are indexed by the elements of a set I is to the confusion of the reader abbreviated to $\{a_i\}$ that, strictly speaking, denotes just the one-element set consisting of a_i .

The number of elements in a set S is called the *cardinality* of S and denoted by $\#(S)$ or $\text{card } S$.

The sets of all integer, nonnegative integer, natural, i.e., **positive** (in some books — not this one — non-negative) integer, rational, real and complex numbers are denoted by \mathbb{Z} , \mathbb{Z}_+ , \mathbb{N} , \mathbb{Q} , \mathbb{R} and \mathbb{C} , respectively.

A 3-gon is called a *triangle*; a 4-gon is called a *quadrilateral*; a 5-gon is called a *pentagon*; a 6-gon is called a *hexagon*; a 7-gon is called a *heptagon*; a 8-gon is called a *octagon*; ¹; a 10-gon is called a *decagon*; etc.

A triangle with nonequal sides is called a *scalene* one; a triangle with two equal sides is called an *isosceles* one. A *regular polygon* is a one with equal sides and equal angles but a triangle with equal sides (and, therefore, angles) is called *equilateral* and never *regular*.

In this book we often abbreviate a *straight line* to *line*; otherwise it is called a *curve*, unless otherwise specified. So a *borderline* or an *airline* is generally a curve while a *broken* line consists of segments of lines. We use the term *segment* speaking of a *line segment*; when other segments appear, e.g., a *spherical segment*, we say so.

A *graph* is a collection of points (*vertices*) connected with curves (*edges*). It often happens that having constructed a graph we distill the objects under the discussion and relations between them. Several theorems of the graph theory are often encountered (*Euler's*, *Hall's*, *Menger's theorems*), cf. solutions to Problems .

A set of points in space are said to be in *general position* if no three of them lie on one line. A set of lines in space are said to be in *general position* if no three of them pass through one point and any two intersect (any three lines form a triangle).

The angle between two curves is the angle between the tangents to these curves at an intersection point. (Obviously, there is a choice among *two* angles; a choice of orientation and order of the curves fixes one of the angles. Often, however, it does not matter which of the angles we choose, i.e., if both angles are right ones.)

A *midperpendicular* to a segment is the line perpendicular to the segment and intersecting it at its center. The ray that bisects an angle, or part of the ray that lies inside the polygon under consideration, is called the *bisector* of the angle. (The term *perpendicular bisector* is often used instead of midperpendicular but not in this book.)

We say that a circle is *inscribed into* or *circumscribed about* a triangle if it is tangent to the triangle's three sides (from the inside of course) or passes through the triangle's vertices, respectively. A circle is called an *escribed about* a triangle if it is tangent, from the outside, to one side and the extensions of the triangle's two other sides.

Two lines (or their segments) in space are said to be *skew* if they do not intersect but are nonparallel; the *angle between skew lines* is the angle between two intersecting lines parallel to the skew lines.

A *dihedral angle* is a spatial angle between two intersecting planes π_1 and π_2 ; it is measured as the angle between two intersecting lines p_1 and p_2 perpendicular to the intersection line of the two planes and such that $p_1 \subset \pi_1$, $p_2 \subset \pi_2$.

A *trihedral angle* is any one of the 8 convex parts of the space between three intersecting planes (with no points of the planes inside it).

A *quadrant* is a quarter of the plane formed by coordinate axes. An *octant* is a trihedral angle with right angles at all its planar angles.

Two figures that can be identified after (1) a parallel translation, or (2) a rotation, or (3) a reflection through a plane or a line and (4) a composition of movements of types (1)–(3) are called *equal*².

A *great circle* on a sphere is a one obtained by intersection of the sphere with a plane passing through the sphere's center.

¹It is funny, no medium-sized dictionary contains a word with a Greek or Latin root for a 9-gon. In this book 9-gons are discriminated, too: they never appear.

²Lately the books on elementary mathematics made the life of many a student and their instructors much more thrilling by introducing the terms *equivalent* for figures of equal area and *congruent* for figures that can be identified after an orientation-preserving transformation. The usage of the fancy term “congruent” would have been OK if it were not that much at variance with our every day usage of the language and selfcontradictory at times.

An angle with vertex at the center of a circle is often measured in *radians* or *degrees*, same as the arc it subtends on the circle, so the notation of the form $\angle A = \frac{1}{2}\sphericalangle BC$ makes sense.

(a, b) denotes the set of real numbers x such that $a \leq x < b$; we similarly define $[a, b]$, $(a, b]$, etc. We sometimes write $]a, b[$ for (a, b) , $]a, b]$ for $(a, b]$, etc. Observe that either of a and b here might be equal to $-\infty$ or ∞ .

$[x]$ denotes the *integer part* of x , i.e., the greater integer that does not exceed x , e.g. $[5] = 5$, $[1\frac{1}{2}] = 1$, $[3/4] = 0$, $[3/2] = 1$, $[\pi] = 3$, $[-1.5] = -2$, etc.

$\{x\} = x - [x]$ denotes the *fractional part* of x .

$n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot (n-1) \cdot n$; this reads *factorial* (of) n . Clearly, $1! = 1$; we convene that $0! = 1$.

$(2n)!! = 2 \cdot 4 \cdot 6 \cdot \dots \cdot (2n-2) \cdot 2n$ denotes the product of n consecutive even numbers (reads *semi-factorial of $2n$*).

$(2n-1)!! = 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-3) \cdot (2n-1)$ denotes the product of n consecutive odd numbers (reads *semi-factorial of $2n-1$*).

Clearly, $(2n)!! \cdot (2n-1)!! = (2n)!$

The inverse functions \sin^{-1} , \cos^{-1} , etc. were denoted in the USSR and in many older books by *arcsin*, *arccos*, etc. The oldfashioned notation has an advantage: no chance to confuse (except by accident) the value of the inverse function at a point with the reciprocal of the value of the function, e.g., generally, $\arcsin(x) \neq (\sin(x))^{-1}$.

Recall that

$$\begin{aligned} x = \arcsin y &\iff |y| \leq 1, \quad |x| \leq \frac{\pi}{2}; \quad y = \sin x \\ z = \arccos t &\iff |t| \leq 1, \quad |z| \in [0, \pi]; \quad t = \cos z. \end{aligned}$$

$\lg x$ stands for $\log_{10} x$ and $\ln x$ or $\log x$ for the *natural* logarithm with base

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = 2.71828\dots$$

A periodic decimal fraction is usually abbreviated to

$$a_n \dots a_1 a_0 . a_{-1} \dots a_{-m} (a_{-m-1} \dots a_{-m-t}),$$

where the digits in parentheses constitute the least period (of length t).

$a^{-n} = \frac{1}{a^n}$ for a positive integer n .

Two numbers a and b are called *incommensurable* if $\frac{a}{b} \notin \mathbb{Q}$.

Facts from algebra. An integer a is said to be *divisible* by an integer b if $a = bc$ for some integer c ; in this case a is a *multiple* of b (and c) and b is a *divisor* of a ; this is sometimes denoted as $a \dot{:} b$ or $b | a$, e.g. $a \dot{:} 2$ if and only if a is an even number. A *proper divisor* of a is an integer divisor b such that $1 < b < |a|$. A *prime* number is an integer $p > 1$ without proper divisors.

Let a and b be positive integers; we denote by $GCD(a, b)$ or just (a, b) for brevity their *greatest common divisor*, i.e., the maximal positive integer c such that both a and b are divisible by c . We denote by $LCM(a, b)$ their *least common multiple*, the least positive integer divisible by both a and b . The following property of GCD and LCM is often used to calculate LCM : $GCD(a, b) \times LCM(a, b) = ab$.

The above definition of $GCD(a, b)$ can be used to define $GCD(a, b)$ for $a, b \in \mathbb{Z}$ when $b \neq 0$; the above definition of $LCM(a, b)$ fits any nonzero integers; these generalized notions satisfy

$$GCD(a, b) \times LCM(a, b) = |ab|.$$

Numbers a and b such that $(a, b) = 1$ are called *relatively prime* or *coprime* if $GCD(a, b) = 1$; if a and b are relatively prime then $ac \dot{:} b$ implies $c \dot{:} b$.

* * *

We use divisibility theorems to prove by the rule of contraries the existence of irrational numbers, e.g., $\sqrt{2}$, $\sqrt{3}$ are irrational. (The same idea applies to \sqrt{q} and \sqrt{pq} for prime p, q .)

The fundamental theorem of arithmetic. Every integer $n > 1$ is the product of its prime divisors defined uniquely up to a permutation. In particular, if $p_1 < p_2 < \dots < p_s$ are all divisors of an integer n , then the representation

$$n = p_1^{k_1} p_2^{k_2} \dots p_s^{k_s}, \quad \text{where } k_1, \dots, k_s \in \mathbb{Z}_+,$$

always exists and is unique.

An integer a is said to be *divisible* by a nonzero integer b with *remainder* r if

$$a = bq + r \quad \text{for some integers } q \text{ and } r, \quad 0 \leq r < |b|.$$

We sometimes use the notation $r = r(a)$ for a given b . For a fixed b the possible values of $r(a)$ are $0, 1, \dots, b - 1$ and are called *residues modulo* b . Two numbers x and y are *congruent modulo* m if $(x - y) \div m$. We write $x \equiv y \pmod{m}$.

It is easy to demonstrate that if $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$ then $a + c \equiv b + d \pmod{n}$ and $ac \equiv bd \pmod{n}$.

Similarly, given polynomials $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ and $g(x) = b_m x^m + b_{m-1} x^{m-1} + \dots + b_1 x + b_0$, we say that $f(x)$ is *divisible by* $g(x)$ if $f(x) = g(x)q(x)$ for some polynomial $q(x)$.

Recall, that the *degree* of a polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ is the greatest power of its nonzero monomial; we write $\deg f(x) = n$. If $g(x) \neq 0$ it is always possible to uniquely represent $f(x)$ in the form

$$f(x) = g(x)q(x) + r(x), \quad \text{where } \deg r(x) < \deg g(x).$$

The above formula implies

Bezout' theorem: For any number x_0 a polynomial $f(x)$ can be represented in the form

$$f(x) = q(x)(x - x_0) + r(x), \quad \text{where } q(x) \text{ is a polynomial.}$$

Proof. In the displayed formula take $g(x) = x - x_0$, $\deg r(x) < \deg(x - x_0) = 1$; hence, $r(x)$ is a constant function: $r(x) = r(x_0)$, as was required. Q.E.D.

If $f(x_0) = 0$, then x_0 is a *root* or a *zero* of the polynomial f ; this is true if and only if f is divisible by $x - x_0$.

The fundamental theorem of algebra. Every nonconstant polynomial $f(x)$ with complex coefficients has $(\deg f)$ complex roots.

Inequalities. *Cauchy's inequality*¹

$$\frac{a_1 + a_2 + \dots + a_n}{n} \geq \sqrt[n]{a_1 \dots a_n} \quad \text{for } a_1 \geq 0, a_2 \geq 0, \dots, a_n \geq 0$$

relates the *arithmetic mean* (the lhs) and the *geometric mean* (the rhs). One can prove it by induction (rather tedious job). A particular case is the relation (prove it!):

$$\frac{a_1 + a_2}{2} \geq \sqrt{a_1 \cdot a_2} \geq \frac{2}{\frac{1}{a_1} + \frac{1}{a_2}},$$

where the last term is the *harmonic mean*.

Though we will not use it in this book, it is too tempting not to mention here the following fact (prove it yourself). Denote by $S_p(a, b) = \left(\frac{a^p + b^p}{2}\right)^{\frac{1}{p}}$ for any non-negative real a, b and $p \neq 0$ the *p-th order mean of a and b*. On Fig. 1 the following proposition is illustrated:

If $a < b$ then $a \leq S_p(a, b) \leq S_q(a, b) \leq b$ for any $p \leq q$.

FIGURE 1. (N1)

FIGURE 2. (N2)

The means of order p and $-p$ are related: $S_p(a, b)S_{-p}(a, b) = ab$. One can prove (do it!) that $\lim_{n \rightarrow \infty} S_{\frac{1}{n}}(a, b) = \sqrt{ab}$ and, therefore, it is natural to *define* $S_0(a, b)$ as \sqrt{ab} .

Progressions. An *arithmetic progression* is a sequence $\{x_n, \text{ where } n \in \mathbb{N}\}$ in which $x_n = x_{n-1} + d$. Hence, $x_n = x_0 + nd$.

Example. The progression $x_n = n$ (hence, $d = 1$) is often referred to as the *natural series*.

¹Sometimes called *Schwarz' inequality* in Germany and *Bounyakovsky's inequality* in Russia.

For an arithmetic progression $\{x_n\}_{n \in \mathbb{N}}$ we have (add $x_0 + \dots + x_n$ with $x_n + \dots + x_0$ term-wise):

$$\sum_{k=0}^n x_k = \frac{x_0 + x_n}{2}(n+1).$$

A *geometric progression* is a sequence of nonzero terms $\{x_n\}_{n \in \mathbb{N}}$ in which $x_n = x_{n-1}q$. Hence, $x_n = q^n \cdot x_0$. An example: $x_n = q^n$. For a geometric progression $\{x_n\}_{n \in \mathbb{N}}$ with $q \neq 1$ we have:

$$\sum_{k=0}^n x_k = x_0 \frac{1 - q^{n+1}}{1 - q}.$$

If $|q| < 1$, then $|q^n|$ tends to 0 as $n \rightarrow \infty$; hence, we can define the infinite sum of all terms of the geometric progression to be $\sum_{k=0}^{\infty} x_k = x_0 \frac{1}{1 - q}$.

A *Fibonacci sequence* is a sequence $\{x_n\}_{n \in \mathbb{N}}$ in which $x_n = x_{n-1} + x_{n-2}$. Sometimes n is allowed to run over \mathbb{Z} . Most often we encounter the sequence with $x_0 = 0$, $x_1 = 1$, i.e.,

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \dots$$

The rule of sum and the rule of product. Let A and B be finite sets, $n \geq 1$ and $m \geq 1$ be their cardinalities, respectively.

S) If A and B have no common elements then there are exactly $n + m$ elements contained in the union of these sets.

P) There are exactly nm ordered pairs (a, b) with $a \in A$, $b \in B$.

The rule of product enables one to calculate C_n^k — the number of ways to choose k elements from n given (indistinguishable) ones. The answer: $C_n^k = \frac{n!}{k!(n-k)!}$. Another common notation for this number is $\binom{n}{k}$, reads: n choose k .

For any numbers a , b and a nonnegative integer n we have the *binomial formula*:

$$(a + b)^n = \sum_{k=0}^n C_n^k a^k b^{n-k}.$$

Observe that $C_n^k = C_n^{n-k}$ and deduce important identities: $\sum_{k=0}^n C_n^k = (1 + 1)^n = 2^n$ and $\sum_{k=0}^n (-1)^k C_n^k = 0$.

Viète's theorem. The roots x_1 , x_2 of a quadratic equation $ax^2 + bx + c = 0$ satisfy the following relations: $x_1 + x_2 = -b/a$, $x_1 \cdot x_2 = c/a$.

Facts from geometry. A *midline* of a triangle (the *midline* of a trapezoid) is the line segment connecting midpoints of two sides of the triangle (trapezoid). The midline's characteristic property (prove it!): *the midline is equal to a half the third side of the triangle (a half sum of the base and the upper side of the trapezoid).*

The *diameter* of a set in space (plane, line) is the maximum (more exactly, the least upper bound) of distances between every pair of its points.

A figure is called *convex* if together with any pair of its points it contains the segment that connects them. The *convex hull* of a set is the figure formed by segments that connect every pair of points of the given set.

Any *polygon* is assumed to be non-selfintersecting and convex unless otherwise specified.

For a triangle ABC with sides a , b , c opposite angles A , B , C , respectively, the *height*, the *bisector* and the *median* dropped from the vertex with angle A onto side a (or its continuation) is denoted by h_a , l_a and m_a . Similar notations are used for the other angles.

We often denote by r and R the (length of the) radii of the inscribed and the circumscribed circles, respectively.

The *inner* and the *outer* tangents to two circles on the plane are those of the form plotted on Fig. 2 and denoted by t_{in} and t_{out} , respectively.

The *orthocenter* of a triangle is the intersection point of the triangle's heights.

The law of sines: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$.

The law of cosines: $c^2 = a^2 + b^2 - 2ab \cos C$.

We will often denote the area of a polygon P by S_P .

Formulas for calculating the area of a triangle ABC :

$$S_{ABC} = \frac{1}{2}ah_a = \frac{1}{2}ab \sin C = \sqrt{p(p-a)(p-b)(p-c)},$$

where $p = \frac{1}{2}(a+b+c)$ (often denoted by s) is the *semiperimeter*. The last formula (with the square root) is called *Heron's formula*.

Thales' theorem. *On the legs of an angle parallel straight lines intercept the segments whose lengths satisfy: $a : b : c = a' : b' : c'$, cf. Fig. 3.*

FIGURE 3. (N3)

FIGURE 4. (N4)

Theorem on medians. *The three medians of a triangle meet at one point. This point divides every median into two segments with the ratio of their lengths 2 : 1 (counting from the corresponding vertex).*

Theorem on bisectors. *All three bisectors of a triangle meet at one point — the center of the inscribed circle.*

Theorem on a bisector. *The bisector of the internal angle C of a triangle ABC divides the opposite side c into segments a' and b' , adjacent to the sides a and b , respectively, so that $a' : b' = a : b$.*

Theorem on midperpendiculars. *The three midperpendiculars of a triangle meet at one point — the center of the circumscribed circle.*

Theorem on heights. *The three heights of a triangle meet at one point — the center of the circumscribed circle for the triangle on whose sides lie the vertices A , B , C and which are parallel to the corresponding sides of $\triangle ABC$, see Fig. 4. The intersection point of heights is called the *orthocenter* of $\triangle ABC$.*

Criteria for two triangles to be equal. *Two triangles ABC and $A'B'C'$ are equal if and only if any of the following is satisfied:*

- 1) $|AB| = |A'B'|$, $|AC| = |A'C'|$ and $\angle A = \angle A'$;
- 2) $|AB| = |A'B'|$, $\angle A = \angle A'$, and $\angle B = \angle B'$;
- 3) $a = a'$, $b = b'$, $c = c'$.

The measure of a central angle in a circle is equal to the measure of the base arc this angle intercepts. The measure of the inscribed angle in a circle is equal to half the measure of the base arc this angle intercepts, see Fig. 4.

Theorem on three perpendiculars. *Let k and l be two straight lines such that a plane Π contains k and the projection m of l on Π is a straight line. Then $(k \perp l)$ if and only if $k \perp m$.*

Instruments you can use to draw figures on a plane. *Calipers* allow one to measure the distance between any two points and find a point on a previously drawn line at a given distance from some point on that line. Unlike *compasses*, they do not let you draw a circle. A *compass* is used to draw a circle of any given radius around a fixed point on a plane and on the surface of a sphere. (The radius of the circle on the sphere is unknown.)

A *one-sided ruler* allows one to draw straight lines; a *two-sided ruler* enables us to draw parallel lines with the distance between them equal to the width of the ruler. These rulers are like a regular ruler but without marks.

A *protractor* is used to translate any given angle on a plane in such a way that one of the legs of the angle assumes any given position.

Miscellanea. Let i be the imaginary unit, i.e., $i^2 = -1$. *Euler's formula* holds:

$$e^{i\varphi} = \cos \varphi + i \sin \varphi. \quad (\text{E})$$

This remarkable formula is the only one worth memorizing in the whole of trigonometry: since for any complex numbers z and w we have $e^z e^w = e^{z+w}$, the reader will quickly learn to use (E) to derive in no time the facts like $\sin(a+b) = \sin a \cos b + \sin b \cos a$.

The *inner* (or *scalar*) *product* of two nonzero vectors \mathbf{a} and \mathbf{b} , denoted by $\mathbf{a} \cdot \mathbf{b}$ or by (\mathbf{a}, \mathbf{b}) , is defined as $|\mathbf{a}||\mathbf{b}| \cos \varphi$, where φ is the angle between \mathbf{a} and \mathbf{b} . If (a_1, a_2, a_3) and (b_1, b_2, b_3) are Cartesian coordinates of \mathbf{a} and \mathbf{b} , then $(\mathbf{a}, \mathbf{b}) = a_1 b_1 + a_2 b_2 + a_3 b_3$.

If either \mathbf{a} or \mathbf{b} is $\mathbf{0}$, we set $(\mathbf{a}, \mathbf{b}) = 0$.

Solutions of a couple of problems require some topology. In the majority of problems, where some notions from topology seem to be needed, the answer can, actually, be guessed regardless, with the help of common sense, often called by mathematicians *physical considerations*. For example, when talking about a line segment, it is often inessential for the answer to be derived whether the segment's endpoints belong to it or not; we consider it not as a set, but as a structureless solid.

Still, in several problems it helps to know that any open interval (a, b) , the union of any number of intervals and the intersection of any *finite* number of intervals is called an *open set* on the line. If a point P belongs to an open interval, this interval is called its *open neighborhood*. A point $P \in M$ is called an *inner* one for the set M if there is an open neighborhood of P that belongs to M . For example, every point of interval (a, b) is inner (prove it!). A point $P \in M$ is called an *outer* one for the set M if there is an open neighborhood of P that does not belong to M . A point that is neither inner nor outer is called a *boundary* one.

On the plane, the open discs $\{x, y \mid (x-a)^2 + (y-b)^2 < r^2\}$ play the role of open intervals in the above definition. In space the discs are replaced with open balls $\{x, y, z \mid (x-a)^2 + (y-b)^2 + (z-c)^2 < r^2\}$.

Observe that an open interval considered not on the line but on the plane or in space does not consist anymore of inner points: the open sets have changed.

The end of the proof is sometimes marked with a \square or Q.E.D.

Selected lectures of mathmathematics circles

Dirichlet's principle

(Summary of Acad. I. M. Gelfand's lecture for 9-th –11-th graders)

First, Gelfand proposed the following

Problem: *An infinite number of narrow parallel ditches $\sqrt{2}$ apart have been dug out across a very long straight road, see Fig. L1. Prove that no matter how narrow these ditches are, a pedestrian with a step of length 1 m inevitably steps into a ditch.*

FIGURE 1. (Fig.L1)

FIGURE 2. (Fig.L2)

A short proof follows from *Dirichlet's* or *pigeonhole principle*.

Indeed, suppose we can 'wind' the road onto a reel of circumference $\sqrt{2}$ m, see Fig. L2. Then all ditches coincide and every step of the pedestrian is marked on the circle by an arc 1 m long. Let us mark the pedestrian's traces (points where the pedestrian touched the ground) after each step. We must prove that at least one of these traces belongs to the interior of a small arc on the circle representing the ditches no matter how short length h of this arc may be.

It is easy to see that if it is possible to find k and l such that the distance between the traces of the k -th step and $(k+l)$ -th step along the circle is less than h , then the desired statement will be easy to prove.

FIGURE 3. (Fig.L3)

Indeed, after l more steps the $(k+2l)$ -th trace moves again by a distance less than h , see Fig. L3; next we consider another l steps, and so on. Now, it is clear that after several groups of l steps we will inevitably discover a trace that falls in a ditch (since by hopping each time the same distance less than h it is impossible to hop over a ditch of width h).

Thus, we should find two traces on the circle with the distance between them less than h . This is where the rabbits (pigeons) come in handy.

Let us divide the circle into arcs of lengths less than h and call these arcs *hutches*. Suppose there are p of them. If we take more than p traces (observe that no two traces coincide since $\sqrt{2}$ is irrational) then by

Dirichlet's principle at least one of the hutches contains more than one trace (rabbit). The distance between two traces that belong to one arc (hutch) is less than h . This proves our statement.

As a second example of the same realm of ideas, consider the following

Problem. *Prove that there exists a power of 2 whose decimal expression begins with three nines, i.e., $2^n = 999\dots$*

In other words, prove that there exist integers n and k such that

$$999 \cdot 10^k \leq 2^n < 10^{k+3}$$

or, equivalently,

$$k + \lg(999) \leq n \lg 2 < k + 3.$$

It is easy to see that this problem is quite similar to the initial one, the only difference being that here the length of the 'step' is equal to $\lg 2$ and that the distance between two neighboring 'ditches' of width $3 - \lg(999)$ is equal to 1.

In general, if p is not a power of 10, then among the numbers p, p^2, p^3, \dots we can find one whose decimal expression begins with any given combination of numbers.

Further elaboration of the same argument leads to a number of interesting theorems of algebra and geometry. Here are some of them:

1) *Let l be a ray originating from a point on the x -axis, $\tan \alpha$ an irrational number, α the angle between the ray and the x -axis. Then l never crosses a point with integer coordinates but passes however close to some of such points.*

2) *There is a positive integer n such that $\sin n < 10^{-10}$.*

3) *If numbers α and β are incommensurable with π and with each other, then for any prescribed distance¹ ε an n can be chosen so that² $|\sin(n\alpha) + \sin(n\beta) - 2| < \varepsilon$ although $\sin(n\alpha) + \sin(n\beta)$ is not equal to 2 for any n .*

4) *If the radii of circles F and G are incommensurable (i.e., their ratio is irrational) then as circle F rolls without slipping along the fixed circle G any point of F traces a curve (called *epicycloid*), see Fig. L4) whose cusps are dense³ on G .*

In conclusion of the lecture Gelfand discussed some qualitative estimates connected with Dirichlet's principle. For example, the problem on the pedestrian striding along the road with ditches was modified as follows: "How often will the pedestrian step into a ditch?"

Nondecimal number systems⁴

(Summary of A. M. Yaglom's lecture for 7-th and 8-th graders)

First, Yaglom challenged the students to play against him the game 'Nim'. This is a game played on the blackboard. Three pieces are placed on the 'chessboard' with three rows, see Fig L4. Each player can move any of the pieces to the right as far as (s)he likes. The winner is the one who makes the last move.

FIGURE 4. (Fig.L4)

Yaglom had prepared a number of winning positions and, using them, easily won several sets on the blackboard, the audience cheering the players. This experiment convinced the students that there existed winning and losing positions; then Yaglom led to the idea to practice playing Nim on a small chessboard.

Further on, Yaglom told the audience about *nondecimal number systems*. Fix a number q . Any positive integer x can be expressed in the form $x = a_n \cdot q^n + a_{n-1} \cdot q^{n-1} + \dots + a_1 \cdot q + q_0$, where $0 \leq a_i < q$. If $q = 10$, we have the standard decimal representation, usually written in the abbreviated form as $x = \overline{a_n a_{n-1} \dots a_1 a_0}$.

¹ ε usually stands among mathematicians for a small number; I wanted to show that it can be very small. *D.L.*

²You should make sure that you *understand* why this means that n can be chosen so that the distance between the points $\sin(n\alpha) + \sin(n\beta)$ and 2 is smaller than ε .

³i.e., any arc of G hosts infinitely many of the cusp points.

⁴For details see [YY].

If $q = 2$, we get the binary system widely used in activities related with computers and coding. With respect to this number system any number is expressed with the help of only two figures, 0 and 1, e.g. $1 = 1_2$, $2 = 10_2$, $4 = 100_2$, $8 = 1000_2$, $9 = 1001_2$, etc. (Here the subscript indicates “the base” of the number system). Fractions can also be written in the same fashion, e.g. $0.101_2 = 1 \cdot 2^{-1} + 0 \cdot 2^{-2} + 1 \cdot 2^{-3}$.

Next, Yaglom said that, more generally, a *number system* is a method to express numbers with the help of a certain “basis” $u_1, u_2, \dots, u_k, \dots$ as follows:

$$N = a_1 \cdot u_1 + a_2 \cdot u_2 + \dots + a_k \cdot u_k + \dots, \quad (*)$$

where the basis need not necessarily be the sequence of powers of a fixed number q and where the k -th “digit” a_k does not exceed a_k/a_{k-1} .

For instance, take for a basis the sequence of *factorials*, i.e., take $u_{k+1} = (k+1)u_k$, $u_0 = 1$. Then any number N is expressed in the form (*), where the k -th “digit” a_k does not exceed $k = a_k/a_{k-1}$, for example

$$1000 = 1 \cdot 720 + 2 \cdot 120 + 1 \cdot 24 + 2 \cdot 6 + 2 \cdot 2 + 0 \cdot 1 = 121220_{Fa}.$$

The *Fibonacci number system* is another example. Its basis is of the form

$$1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \dots \quad (\text{i.e., } u_{k+2} = u_{k+1} + u_k), \dots$$

With respect to this system, any digit is either 0 or 1, as in the binary one, but here *no two 1's can stand in a row*¹, e.g.

$$100 = 1 \cdot 89 + 0 \cdot 55 + 0 \cdot 34 + 0 \cdot 21 + 0 \cdot 13 + 1 \cdot 8 + 0 \cdot 5 + 1 \cdot 3 + 0 \cdot 2 + 0 \cdot 1 = 1000010100_{Fib}.$$

Exercise. 1) *How to pass from one number system to another?*

2) *Write the multiplication table for the above systems.*

3)* *How to express fractions in the Fibonacci system?*

Then Yaglom used the binary system to discuss possible victories and defeats in Nim. Together with the audience they derived the rule:

A position (a, b, c) is a losing one if all sums of digits of all numbers a, b and c in the binary system corresponding to the same position are even.

The lecture ended with an analysis in terms of the Fibonacci system of another game, *zhì shí zi* (reads “tsin shi tsi”) which in Chinese means *throwing stones*; its Western name is *Withoff's game*. Zhì shí zi differs from Nim in that its board has two rows; a player can shift with every move either one chip (to any place) or one can simultaneously shift both chips by the same distance, cf. Fig. L5.

FIGURE 5. (Fig.L5)

Yaglom proved that a position (a, b) , where $a < b$, loses if a_{Fib} ends with an even number of zeros and $b_{Fib} = a0_{Fib}$.

Here is a sketch of a proof.

Positions $([n\tau^2], [n\tau^2])$, where $n = 1, 2, \dots$ and $\tau \approx 1.618 \dots$ is the positive root of the quadratic equation $x^2 = x + 1$, win.

Proof is based on the following lemma.

Lemma. *Let X, Y be positive irrational numbers such that*

$$\frac{1}{X} + \frac{1}{Y} = 1. \quad (*)$$

Then every positive integer can be uniquely represented as either $[\frac{k}{X}]$ or $[\frac{l}{Y}]$ for some $k, l \in \mathbb{N}$.

Proof. Among numbers $1, 2, \dots, n$ there are $[\frac{n}{X}]$ numbers of the form $[kX]$ and $[\frac{n}{Y}]$ numbers of the form $[lY]$. Since $\frac{1}{X} + \frac{1}{Y} = 1$, it follows that $\frac{n}{X} + \frac{n}{Y} = n \implies [\frac{n}{X}] + [\frac{n}{Y}] = n - 1$. Similarly, among $1, 2, \dots, n+1$ there are n such numbers. Therefore, between n and $n+1$ there is exactly one such number, Q.E.D.

Now, let us solve the problem. Since τ and τ^2 satisfy relation (*), every natural number is of the form either $[n\tau]$ or $[m\tau^2]$. Moreover, it is clear that

$$[n\tau^2] - [n\tau] = n. \quad (**)$$

Hence, the pairs $([m\tau^2], [m\tau])$ cover the whole natural series, and differences (**) are distinct for distinct m . But this means exactly that we have found the set of losing positions for the first player, Q.E.D.

Can you figure out how to explicitly incorporate the Fibonacci system in the proof?

¹It is rather difficult to explain why! For a solution see [YY].

Indefinite second-order equations
(Summary of CMA B. N. Deloné's lecture for 9-th – 10-th graders)

Deloné began with a short story about indefinite second-order equations for two integer unknowns. The most interesting among them is *Pell's equation*:

$$x^2 - my^2 = 1, \quad (*)$$

where m is a positive integer which is not a perfect square.

Theorem. *Equation (*) has infinitely many solutions.*

To prove this let us take a rectangular coordinate system u, v and consider vectors $\mathbf{a} = (1, 1)$ and $\mathbf{b} = (\sqrt{m}, -\sqrt{m})$. All points M such that $OM = x\mathbf{a} + y\mathbf{b}$, where x, y are integers, form a *lattice* closely related with the properties of equation (*):

If M is one of the points of the lattice, then in coordinate system u, v the coordinates of M are

$$u = x + y\sqrt{m}, \quad v = x - y\sqrt{m}$$

and therefore $uv = x^2 - my^2$.

Thus, the proof of our Theorem reduces to the following

Problem: *Prove that the hyperbola $uv = 1$ contains infinitely many points of the lattice. (The hyperbola is plotted by the dashed curve on Fig. L6.)*

FIGURE 6. (Fig.L6)

FIGURE 7. (Fig.L7)

One point of the lattice belonging to the hyperbola is obvious: the point M_0 with coordinates $u = v = 1$. The symmetric point M'_0 ($u = v = -1$) also belongs to the hyperbola. Suppose that in addition to these two points we have found one more point of the lattice, $M_1(u_1, v_1)$ such that $u_1v_1 = 1$.

Consider the transformation φ of the plane that sends an arbitrary point $A(u, v)$ into $A' = \varphi(A)$ with the coordinates $u' = uu_1, v' = vv_1$. Clearly, φ transforms the hyperbola $uv = 1$ into itself, i.e., the transformation moves the hyperbola along itself (and that is why mathematicians call such φ a *hyperbolic rotation*). Indeed,

$$u'v' = uu_1 \cdot vv_1 = uv \cdot u_1v_1 = 1.$$

Then, it is easy to verify that *the hyperbolic rotations φ map the points of the lattice into points of the lattice.*

Indeed, since M_1 is a point of the lattice, it follows that

$$u_1 = x_1 + y_1\sqrt{m}, \quad v_1 = x_1 - y_1\sqrt{m},$$

where x_1, y_1 are integers. Further on, if

$$M(u, v) = (x + y\sqrt{m}, x - y\sqrt{m})$$

is one more point of the lattice, and x, y are integers, then

$$u' = uu_1 = (x + y\sqrt{m})(x_1 + y_1\sqrt{m}) = (xx_1 + yy_1m) + (xy_1 + x_1y)\sqrt{m} = X + Y\sqrt{m};$$

$$v' = vv_1 = (x - y\sqrt{m})(x_1 - y_1\sqrt{m}) = (xx_1 + yy_1m) - (xy_1 + x_1y)\sqrt{m} = X - Y\sqrt{m},$$

i.e., point $M' = \varphi(M)$ with coordinates (u', v') also belongs to the lattice.

The hyperbolic rotation φ transforms $M_0(1, 1)$ into $M_1(u_1, v_1)$ and M_1 into $M_2 = \varphi(M_1)$, a new point of the lattice belonging to the hyperbola. The same rotation transforms M_2 into $M_3 = \varphi(M_2)$ that also belongs to the hyperbola, etc.

The *inverse rotation* φ^{-1} that transforms (u, v) to point $(u' = \frac{u}{u_1}, v' = \frac{v}{v_1})$ sends M_0 into $M_{-1} = \varphi^{-1}(M_0)$; $M_{-2} = \varphi^{-1}(M_{-1})$, and so on. We get an *infinite* set of points of the lattice

$$\dots, M_{-2}, M_{-1}, M_0, M_1, M_2, \dots$$

which belong to the hyperbola and turn into each other under the hyperbolic rotation φ .

Thus, it suffices to find on the hyperbola at least one point M_1 different from M_0 and M'_0 .

In order to do this let us move the segment connecting points $(1, 1)$ and $(1, -1)$ to the right along the u -axis until it meets a point N' of the lattice. If (u', v') are the coordinates of N' , then $|u'| < 1$ and the rectangle G' with vertices at points $(\pm u', \pm v')$ contains only three points of the lattice: the origin O , the point N' and the point symmetric to N' with respect to O .

Now, let us move the right edge of the rectangle along the u -axis until we encounter a new point $N''(u'', v'')$ of the lattice. Then we may again move the right edge of the rectangle G'' , now with vertices at points $(\pm u'', \pm v'')$, along the u -axis, etc. (see Fig. L7).

An elegant argument ascending to Herman Minkowski enables us to establish that the sequence of areas of the rectangles G', G'', \dots (all these areas are integers) is bounded.

Therefore, among them, there are infinitely many rectangles with the same area. Hence, we can deduce that among the N', N'', \dots there exist two points such that a hyperbolic rotation ψ sending one of them into another maps the lattice into itself. Therefore, ψ transforms M_0 into a different point of the lattice that belongs to the hyperbola $uv = 1$.

MOSCOW MATHEMATICAL OLYMPIADS 1 – 59

Olympiad 1 (1935)

Tour 1.1

Set 1.1.A

1.1.A.1. Find the ratio of two numbers if the ratio of their arithmetic mean to their geometric mean is $25 : 24$.

1.1.A.2. Given the lengths of two sides of a triangle and that of the bisector of the angle between these sides, construct the triangle.

1.1.A.3. The base of a pyramid is an isosceles triangle with the vertex angle α . The pyramid's lateral edges are at angle φ to the base. Find the dihedral angle θ at the edge connecting the pyramid's vertex to that of angle α .

Set 1.1.B

1.1.B.1. A train passes an observer in t_1 sec. At the same speed the train crosses a bridge l m long. It takes the train t_2 sec to cross the bridge from the moment the locomotive drives onto the bridge until the last car leaves it. Find the length and speed of the train.

1.1.B.2. Given three parallel straight lines. Construct a square three of whose vertices belong to these lines.

1.1.B.3. The base of a right pyramid is a quadrilateral whose sides are each of length a . The planar angles at the vertex of the pyramid are equal to the angles between the lateral edges and the base. Find the volume of the pyramid.

Set 1.1.C

1.1.C.1. Find four consecutive terms a, b, c, d of an arithmetic progression and four consecutive terms a_1, b_1, c_1, d_1 of a geometric progression such that $a + a_1 = 27$, $b + b_1 = 27$, $c + c_1 = 39$, and $d + d_1 = 87$.

1.1.C.2. Prove that if the lengths of the sides of a triangle form an arithmetic progression, then the radius of the inscribed circle is one third of one of the heights of the triangle.

1.1.C.3. The height of a truncated cone is equal to the radius of its base. The perimeter of a regular hexagon circumscribing its top is equal to the perimeter of an equilateral triangle inscribed in its base. Find the angle φ between the cone's generating line and its base.

Set 1.1.D

1.1.D.1. Solve the system

$$\begin{cases} x^2 + y^2 - 2z^2 = 2a^2, \\ x + y + 2z = 4(a^2 + 1), \\ z^2 - xy = a^2. \end{cases}$$

1.1.D.2. In $\triangle ABC$, two straight lines drawn from an arbitrary point D on AB are parallel to AC and BC and intersect BC and AC at F and G , respectively. Prove that the sum of the circumferences of the circles circumscribed around $\triangle ADG$ and $\triangle BDF$ is equal to the circumference of the circle circumscribed around $\triangle ABC$.

1.1.D.3. The unfolding of the lateral surface of a cone is a sector of angle 120° . The angles at the base of a pyramid constitute an arithmetic progression with a difference of 15° . The pyramid is inscribed in the cone. Consider a lateral face of the pyramid with the smallest area. Find the angle α between the plane of this face and the base.

Tour 1.2

Set 1.2.A

1.2.A.1. The median, bisector, and height, all originate at the same vertex of a triangle. Given the intersection points of the median, bisector, and height with the circumscribed circle, construct the triangle.

1.2.A.2. Find the locus of points on the surface of a cube that serve as the vertex of the smallest angle that subtends the diagonal.

1.2.A.3. Triangles $\triangle ABC$ and $\triangle A_1B_1C_1$ lie on different planes. Line AB intersects line A_1B_1 ; line BC intersects line B_1C_1 and line CA intersects line C_1A_1 . Prove that either the three lines AA_1 , BB_1 , CC_1 meet at one point or that they are all parallel.

Set 1.2.B

1.2.B.1. How many real solutions does the following system have?

$$\begin{cases} x + y = 2, \\ xy - z^2 = 1. \end{cases}$$

1.2.B.2. Solve the system

$$\begin{cases} x^3 - y^3 = 2b, \\ x^2y - xy^2 = b. \end{cases}$$

1.2.B.3. Evaluate the sum: $1^3 + 3^3 + 5^3 + \dots + (2n - 1)^3$.

Set 1.2.C

1.2.C.1. a) How many distinct ways are there of painting the faces of a cube six different colors? (Colorations are considered distinct if they do not coincide when the cube is rotated.)

b)* How many distinct ways are there of painting the faces of a dodecahedron 12 different colors? (Colorations are considered distinct if they do not coincide when the cube is rotated.)

1.2.C.2. How many ways are there of representing a positive integer n as the sum of three positive integers? Representations which differ only in the order of the summands are considered to be distinct.

1.2.C.3. Denote by $M(a, b, c, \dots, k)$ the least common multiple and by $D(a, b, c, \dots, k)$ the greatest common divisor of a, b, c, \dots, k . Prove that:

a) $M(a, b)D(a, b) = ab$;

b) $\frac{M(a, b, c)D(a, b)D(b, c)D(a, c)}{D(a, b, c)} = abc$.

Olympiad 2 (1936)

Tour 2.1

2.1.1. Find a four-digit perfect square whose first digit is the same as the second, and the third is the same as the fourth.

2.1.2. All rectangles that can be inscribed in an isosceles triangle with two of their vertices on the triangle's base have the same perimeter. Construct the triangle.

2.1.3 (*P. Dirac's problem.*) Represent an arbitrary positive integer as an expression involving only 3 twos and any mathematical signs.

2.1.4. Consider a circle and a point P outside the circle. The angle of given measure with vertex at P subtends a diameter of the circle. Construct the circle's diameter with ruler and compass.

2.1.5. Find 4 consecutive positive integers whose product is 1680.

Tour 2.2

2.2.1. Solve the system:

$$\begin{cases} x + y = a, \\ x^5 + y^5 = b^5. \end{cases}$$

2.2.2. Given an angle less than 180° , and a point M outside the angle. Draw a line through M so that the triangle, whose vertices are the vertex of the angle and the intersection points of its legs with the line drawn, has a given perimeter.

2.2.3. The lengths of a rectangle's sides and of its diagonal are integers. Prove that the area of the rectangle is an integer multiple of 12.

2.2.4. How many ways are there to represent 10^6 as the product of three factors? Factorizations which only differ in the order of the factors are considered to be distinct.

2.2.5. Given three planes and a ball in space. In space, find the number of different ways of placing another ball so that it would be tangent the three given planes and the given ball.

Olympiad 3 (1937)

Tour 3.1

3.1.1. Solve the system:

$$\begin{cases} x + y + z = a, \\ x^2 + y^2 + z^2 = a^2, \\ x^3 + y^3 + z^3 = a^3. \end{cases}$$

3.1.2*. On a plane two points A and B are on the same side of a line. Find point M on the line such that $MA + MB$ is equal to a given length.

3.1.3. Two segments slide along two skew lines. Consider the tetrahedron with vertices at the endpoints of the segments. Prove that the volume of the tetrahedron does not depend on the position of the segments.

Tour 3.2

3.2.1. Given three points that are not on the same straight line. Three circles pass through each pair of the points so that the tangents to the circles at their intersection points are perpendicular to each other. Construct the circles.

3.2.2*. Given a regular dodecahedron. Find how many ways are there to draw a plane through it so that its section of the dodecahedron is a regular hexagon?

3.2.3. Into how many parts can an n -gon be divided by its diagonals if no three diagonals meet at one point?

Olympiad 4 (1938)

Tour 4.1

4.1.? (See footnote 1 to *Historical remarks*.) In space 4 points are given. How many planes equidistant from these points are there? Consider separately (a) the generic case (the points given do not lie on a single plane) and (b) the degenerate cases.

Tour 4.2

4.2.1. The following operation is performed over points O_1, O_2, O_3 and A in space. The point A is reflected with respect to O_1 , the resultant point A_1 is reflected through O_2 , and the resultant point A_2 through O_3 . We get some point A_3 that we will also consecutively reflect through O_1, O_2, O_3 . Prove that the point obtained last coincides with A ; see Fig. 1.

FIGURE 1. (Probl. 4.2.1)

4.2.2. What is the largest number of parts into which n planes can divide space?

4.2.3. Given the base, height and the difference between the angles at the base of a triangle, construct the triangle.

4.2.4. How many positive integers smaller than 1000 and not divisible by 5 and by 7 are there?

Olympiad 5 (1939)

Tour 5.1

5.1.1. Solve the system:

$$\begin{cases} 3xyz - x^3 - y^3 - z^3 = b^3, \\ x + y + z = 2b, \\ x^2 + y^2 - z^2 = b^2. \end{cases}$$

5.1.2. Prove that

$$\cos \frac{2\pi}{5} + \cos \frac{4\pi}{5} = -\frac{1}{2}.$$

5.1.3. Consider points A, B, C . Draw a line through A so that the sum of distances from B and C to this line is equal to the length of a given segment.

5.1.4. Solve the equation $\sqrt{a - \sqrt{a + x}} = x$ for x .

5.1.5. Prove that for any triangle the bisector lies between the median and the height drawn from the same vertex. (See Fig. 2.)

FIGURE 2. (Probl. 5.1.5)

FIGURE 3. (Probl. 5.2.3)

Tour 5.2

5.2.1. Factor $a^{10} + a^5 + 1$ into nonconstant polynomials with integer coefficients.

5.2.2. Let the product of two polynomials of a variable x with integer coefficients be a polynomial with even coefficients not all of which are divisible by 4. Prove that all the coefficients of one of the polynomials are even and that at least one of the coefficients of the other polynomial is odd.

5.2.3. Given two points A and B and a circle, find a point X on the circle so that points C and D at which lines AX and BX intersect the circle are the endpoints of the chord CD parallel to a given line MN . (See Fig. 3.)

5.2.4. Find the remainder after division of $10^{10} + 10^{10^2} + 10^{10^3} + \dots + 10^{10^{10}}$ by 7.

5.2.5. Consider a regular pyramid and a perpendicular to its base at an arbitrary point P . Prove that the sum of the lengths of the segments connecting P to the intersection points of the perpendicular with the planes of the pyramid's faces does not depend on the location of P .

5.2.6. What is the greatest number of parts that 5 spheres can divide the space into?

Olympiad 6 (1940)

Tour 6.1

Grades 7 – 8

6.1.7-8.1. Factor $(b - c)^3 + (c - a)^3 + (a - b)^3$.

6.1.7-8.2. It takes a steamer 5 days to go from Gorky to Astrakhan downstream the Volga river and 7 days upstream from Astrakhan to Gorky. How long will it take for a raft to float downstream from Gorky to Astrakhan?

6.1.7-8.3. How many zeros does $100!$ have at its end in the usual decimal representation?

6.1.7-8.4. Draw a circle that has a given radius R and is tangent to a given line and a given circle. How many solutions does this problem have?

Grades 9 – 10

6.1.9-10.1. Solve the system:

$$\begin{cases} (x^3 + y^3)(x^2 + y^2) = 2b^5, \\ x + y = b. \end{cases}$$

6.1.9-10.2. Consider all positive integers written in a row:

123456789101112131415

Find the 206788-th digit from the left.

6.1.9-10.3. Construct a circle equidistant from four points on a plane. How many solutions are there?

6.1.9-10.4. Given two lines on a plane, find the locus of all points with the difference between the distance to one line and the distance to the other equal to the length of a given segment.

6.1.9-10.5. Find all 3-digit numbers \overline{abc} such that $\overline{abc} = a! + b! + c!$.

Tour 6.2

Grades 7 – 8

6.2.7-8.1. See Problem 2.1.1.

6.2.7-8.2. Points A, B, C are vertices of an equilateral triangle inscribed in a circle. Point D lies on the shorter arc, $\cup AB$ (not $\cup ACB$); see Fig. 4. Prove that $AD + BD = DC$.

FIGURE 4. (Probl. 6.2.7-8.2)

6.2.7-8.3. How does one tile a plane, without gaps or overlappings, with the tiles equal to a given irregular quadrilateral?

6.2.7-8.4. How many pairs of integers x, y are there between 1 and 1000 such that $x^2 + y^2$ is divisible by 49?

Grades 9 – 10

6.2.9-10.1*. Given an infinite cone. The measure of its unfolding's angle is equal to α . A curve on the cone is represented on any unfolding by the union of line segments. Find the number of the curve's self-intersections.

6.2.9-10.2. Which is greater: $300!$ or 100^{300} ?

6.2.9-10.3. The center of the circle circumscribing $\triangle ABC$ is mirrored through each side of the triangle and three points are obtained: O_1, O_2, O_3 . Reconstruct $\triangle ABC$ from O_1, O_2, O_3 if everything else is erased.

6.2.9-10.4. Let a_1, \dots, a_n be positive numbers. Prove the inequality:

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \frac{a_3}{a_4} + \dots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1} \geq n.$$

6.2.9-10.5. How many positive integers x less than 10 000 are there such that $2^x - x^2$ is divisible by 7 ?

Olympiad 7 (1941)

Tour 7.1

Grades 7 – 8

7.1.7-8.1. Construct a triangle given its height and median — both from the same vertex — and the radius of the circumscribed circle.

7.1.7-8.2. Find the number $\overline{523abc}$ divisible by 7, 8 and 9.

7.1.7-8.3. Given a quadrilateral, the midpoints A, B, C, D of its consecutive sides, and the midpoints of its diagonals, P and Q . Prove that $\triangle BCP = \triangle ADQ$.

7.1.7-8.4. A point P lies outside a circle. Consider all possible lines drawn through P so that they intersect the circle. Find the locus of the midpoints of the chords — segments the circle intercepts on these lines.

7.1.7-8.5. Prove that 1 plus the product of any four consecutive integers is a perfect square.

Grades 9 – 10

7.1.9-10.1. See Problem 7.1.7-8.2.

7.1.9-10.2. On the sides of a parallelogram, squares are constructed outwards. Prove that the centers of these squares are vertices of a square.

7.1.9-10.3. A polynomial $P(x)$ with integer coefficients takes odd values at $x = 0$ and $x = 1$. Prove that $P(x)$ has no integer roots.

7.1.9-10.4. Given points M and N , the bases of heights AM and BN of $\triangle ABC$ and the line to which the side AB belongs. Construct $\triangle ABC$.

7.1.9-10.5. Solve the equation:

$$|x + 1| - |x| + 3|x - 1| - 2|x - 2| = x + 2.$$

7.1.9-10.6. How many roots does equation $\sin x = \frac{x}{100}$ have?

Tour 7.2

Grades 7 – 8

7.2.7-8.1. Prove that it is impossible to divide a rectangle into five squares of distinct sizes. (Cf. Problem 7.2.9-10.1.)

7.2.7-8.2*. Given $\triangle ABC$, divide it into the minimal number of parts so that after being flipped over these parts can constitute the same $\triangle ABC$.

7.2.7-8.3. Consider $\triangle ABC$ and a point M inside it. We move M parallel to BC until M meets CA , then parallel to AB until it meets BC , then parallel to CA , and so on. Prove that M traverses a self-intersecting closed broken line and find the number of its straight segments.

7.2.7-8.4. Find an integer a for which $(x - a)(x - 10) + 1$ factors in the product $(x + b)(x + c)$ with integers b and c .

7.2.7-8.5. Prove that the remainder after division of the square of any prime $p > 3$ by 12 is equal to 1.

7.2.7-8.6. Given three points H_1, H_2, H_3 on a plane. The points are the reflections of the intersection point of the heights of the triangle $\triangle ABC$ through its sides. Construct $\triangle ABC$.

Grades 9 – 10

7.2.9-10.1. Prove that it is impossible to divide a rectangle into six squares of distinct sizes.

7.2.9-10.2. On a plane, several points are chosen so that a disc of radius 1 can cover every 3 of them. Prove that a disc of radius 1 can cover all the points.

7.2.9-10.3. Find nonzero and nonequal integers a, b, c so that $x(x-a)(x-b)(x-c) + 1$ factors into the product of two polynomials with integer coefficients.

7.2.9-10.4. Solve in integers the equation

$$x + y = x^2 - xy + y^2.$$

7.2.9-10.5. Given two skew perpendicular lines in space, find the set of the midpoints of all segments of given length with the endpoints on these lines.

7.2.9-10.6. Construct a right triangle, given two medians drawn to its legs.

Olympiad 8 (1945)

Tour 8.1

Grades 7 – 8

8.1.7-8.1. Divide $a^{27} - b^{27}$ by $(a+b)(a^2+b^2)(a^4+b^4)\dots(a^{26}+b^{26})$. (Cf. Problem 8.1.9-10.1).

8.1.7-8.2. Prove that for any positive integer n the following inequality holds:

$$\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n} > \frac{1}{2}.$$

8.1.7-8.3. Find all two-digit numbers \overline{ab} such that $\overline{ab} + \overline{ba}$ is a perfect square.

8.1.7-8.4. Prove that it is impossible to divide a scalene triangle into two equal triangles.

8.1.7-8.5. Two circles are tangent externally at one point. Common external tangents are drawn to them and the tangent points are connected. Prove that the sum of the lengths of the opposite sides of the quadrilateral obtained are equal.

Grades 9 – 10

8.1.9-10.1. Divide $a^{2k} - b^{2k}$ by $(a+b)(a^2+b^2)(a^4+b^4)\dots(a^{2k-1}+b^{2k-1})$. (See Problem 8.1.7-8.2.)

8.1.9-10.2. Find three-digit numbers such that any its positive integer power ends with the same three digits and in the same order.

8.1.9-10.3. The system

$$\begin{cases} x^2 - y^2 = 0, \\ (x-a)^2 + y^2 = 1 \end{cases}$$

generally has four solutions. For which a the number of solutions of the system is equal to three or two?

8.1.9-10.4. A right triangle ABC moves along the plane so that the vertices B and C of the triangle's acute angles slide along the sides of a given right angle. Prove that point A fills in a line segment and find its length.

Tour 8.2

Grades 7 – 8

8.2.7-8.1. Given the 6 digits: 0, 1, 2, 3, 4, 5. Find the sum of all even four-digit numbers which can be expressed with the help of these figures (the same figure can be repeated).

8.2.7-8.2. Suppose we have two identical cardboard polygons. We placed one polygon upon the other one and aligned. Then we pierced polygons with a pin at a point. Then we turned one of the polygons around this pin by $25^\circ 30'$. It turned out that the polygons coincided (aligned again). What is the minimal possible number of sides of the polygons?

8.2.7-8.3. The side AD of a parallelogram $ABCD$ is divided into n equal segments. The nearest to A division point P is connected with B . Prove that line BP intersects the diagonal AC at point Q such that $AQ = \frac{AC}{n+1}$; see Fig. 5.

8.2.7-8.4. Segments connect vertices A, B, C of $\triangle ABC$ with respective points A_1, B_1, C_1 on the opposite sides of the triangle. Prove that the midpoints of segments AA_1, BB_1, CC_1 do not belong to one straight line.

Grades 9 – 10

8.2.9-10.1. Solve in integers the equation

$$xy + 3x - 5y = -3.$$

FIGURE 5. (Probl. 8.2.7-8.3)

FIGURE 6. (Probl. 9.1.7-8.2)

8.2.9-10.2. The numbers a_1, a_2, \dots, a_n are equal to 1 or -1 . Prove that

$$2 \sin \left(a_1 + \frac{a_1 a_2}{2} + \frac{a_1 a_2 a_3}{4} + \dots + \frac{a_1 a_2 \dots a_n}{2^{n-1}} \right) \frac{\pi}{4} = a_1 \sqrt{2 + a_2 \sqrt{2 + a_3 \sqrt{2 + \dots + a_n \sqrt{2}}}}$$

In particular, for $a_1 = a_2 = \dots = a_n = 1$ we have

$$2 \sin \left(1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^{n-1}} \right) \frac{\pi}{4} = 2 \cos \frac{\pi}{2^{n+1}} = \sqrt{2 + \sqrt{2 + \dots + \sqrt{2}}}$$

8.2.9-10.3. A circle rolls along a side of an equilateral triangle. The radius of the circle is equal to the height of the triangle. Prove that the measure of the arc intercepted by the sides of the triangle on this circle is equal to 60° at all times.

Olympiad 9 (1946)

Tour 9.1

Grades 7 – 8

9.1.7-8.1. What is the largest number of acute angles that a convex polygon can have?

9.1.7-8.2. Given points A, B, C on a line, equilateral triangles ABC_1 and BCA_1 constructed on segments AB and BC , and midpoints M and N of AA_1 and CC_1 , respectively. Prove that $\triangle BMN$ is equilateral. (We assume that B lies between A and C , and points A_1 and C_1 lie on the same side of line AB , see Fig. 6.)

9.1.7-8.3. Find a four-digit number such that the remainders after its division by 131 and 132 are 112 and 98, respectively.

9.1.7-8.4. Solve the system of equations:

$$\begin{cases} x_1 + x_2 + x_3 = 6, \\ x_2 + x_3 + x_4 = 9, \\ x_3 + x_4 + x_5 = 3, \\ x_4 + x_5 + x_6 = -3, \\ x_5 + x_6 + x_7 = -9, \\ x_6 + x_7 + x_8 = -6, \\ x_7 + x_8 + x_1 = -2, \\ x_8 + x_1 + x_2 = 2. \end{cases}$$

9.1.7-8.5. Prove that after completing the multiplication and collecting the terms

$$(1 - x + x^2 - x^3 + \dots - x^{99} + x^{100})(1 + x + x^2 + \dots + x^{99} + x^{100})$$

has no monomials of odd degree.

Grades 9 – 10

9.1.9-10.1. Given two intersecting planes α and β and a point A on the line of their intersection. Prove that of all lines belonging to α and passing through A the line which is perpendicular to the intersection line of α and β forms the greatest angle with β .

9.1.9-10.2. Through a point M inside an angle a line is drawn. It cuts off this angle a triangle of the least possible area. Prove that M is the midpoint of the segment on this line that the angle intercepts.

9.1.9-10.3. Prove that $n^2 + 3n + 5$ is not divisible by 121 for any positive integer n .

9.1.9-10.4. Prove that for any positive integer n the following identity holds

$$\frac{(2n)!}{n!} = 2^n(2n-1)!!.$$

9.1.9-10.5. Prove that if α and β are acute angles and $\alpha < \beta$, then

$$\frac{\tan \alpha}{\alpha} < \frac{\tan \beta}{\beta}.$$

Tour 9.2

Grades 7 – 8

9.2.7-8.1. Two seventh graders and several eighth graders take part in a chess tournament. The two seventh graders together scored eight points. The scores of eighth graders are equal. How many eighth graders took part in the tournament?

9.2.7-8.2. Prove that for any integers x and y we have:

$$x^5 + 3x^4y - 5x^3y^2 - 15x^2y^3 + 4xy^4 + 12y^5 \neq 33.$$

9.2.7-8.3. On the legs of $\angle AOB$, the segments OA and OB lie; $OA > OB$. Points M and N on lines OA and OB , respectively, are such that $AM = BN = x$. Find x for which the length of MN is minimal.

9.2.7-8.4. Towns A_1, A_2, \dots, A_{30} lie on line MN . The distances between the consecutive towns are equal. Each of the towns is the point of origin of a straight highway. The highways are on the same side of MN and form the following angles with it:

No.	1	2	3	4	5	6	7	8	9	10
	60°	30°	15°	20°	155°	45°	10°	35°	140°	50°
No.	11	12	13	14	15	16	17	18	19	20
	125°	65°	85°	86°	80°	75°	78°	115°	95°	25°
No.	21	22	23	24	25	26	27	28	29	30
	28°	158°	30°	25°	5°	15°	160°	170°	20°	158°

Thirty cars start simultaneously from these towns along the highway at the same constant speed. Each intersection has a gate. As soon as the first (in time, not in number) car passes the intersection the gate closes and blocks the way for all other cars approaching this intersection. Which cars will pass all intersections and which will be stopped?

9.2.7-8.5. A bus network is organized so that:

- 1) one can reach any stop from any other stop without changing buses;
- 2) every pair of routes has a single stop at which one can change buses;
- 3) each route has exactly three stops?

How many bus routes are there?

Grades 9 – 10

9.2.9-10.1. Ninth and tenth graders participated in a chess tournament. There were ten times as many tenth graders as ninth graders. The total score of tenth graders was 4.5 times that of the ninth graders. What was the ninth graders score?

9.2.9-10.2. Given the Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, \dots$, ascertain whether among its first 100 000 001 terms there is a number that ends with four zeros.

9.2.9-10.3. On the sides PQ, QR, RP of $\triangle PQR$ segments AB, CD, EF are drawn. Given a point S_0 inside triangle $\triangle PQR$, find the locus of points S for which the sum of the areas of triangles $\triangle SAB, \triangle SCD$ and $\triangle SEF$ is equal to the sum of the areas of triangles $\triangle S_0AB, \triangle S_0CD, \triangle S_0EF$.

Consider separately the case $\frac{AB}{PQ} = \frac{CD}{QR} = \frac{EF}{RP}$.

9.2.9-10.4. A town has 57 bus routes. How many stops does each route have if it is known that

- 1) one can reach any stop from any other stop without changing buses;
- 2) for every pair of routes there is a single stop where one can change buses;
- 3) each route has three or more stops?

9.2.9-10.5. See Problem 9.2.7-8.4.

Olympiad 10 (1947)

Tour 10.1

Grades 7 – 8

10.1.7-8.1. Find the remainder after division of the polynomial $x + x^3 + x^9 + x^{27} + x^{81} + x^{243}$ by $x - 1$.

10.1.7-8.2. Prove that of 9 consecutive positive integers one that is relatively prime with the others can always be selected.

10.1.7-8.3. Find the coefficients of x^{17} and x^{18} after expansion and collecting the terms of $(1+x^5+x^7)^{20}$.

10.1.7-8.4. Given a convex pentagon $ABCDE$, prove that if an arbitrary point M inside the pentagon is connected by lines with all the pentagon's vertices, then either one or three or five of these lines cross the sides of the pentagon opposite the vertices they pass.

10.1.7-8.5. Point O is the intersection point of the heights of an acute triangle $\triangle ABC$. Prove that the three circles which pass: a) through O, A, B , b) through O, B, C , and c) through O, C, A , are equal. (See Fig. 7.)

FIGURE 7. (Probl. 10.1.7-8.5)

Grades 9 – 10

10.1.9-10.1. Find the coefficient of x^2 after expansion and collecting the terms of the following expression (there are k pairs of parentheses):

$$((\dots(((x-2)^2-2)^2-2)^2-\dots-2)^2-2)^2.$$

10.1.9-10.2. See Problem 10.1.7-8.2 for 16 consecutive numbers.

10.1.9-10.3. How many squares different in size or location can be drawn on an 8×8 chess board? Each square drawn must consist of whole chess board's squares.

10.1.9-10.4. Which of the polynomials, $(1+x^2-x^3)^{1000}$ or $(1-x^2+x^3)^{1000}$, has the greater coefficient of x^{20} after expansion and collecting the terms?

10.1.9-10.5. Calculate (without calculators, tables, etc.) with accuracy to 0.00001 the product

$$\left(1 - \frac{1}{10}\right) \left(1 - \frac{1}{10^2}\right) \dots \left(1 - \frac{1}{10^{99}}\right).$$

10.1.9-10.6. Given line AB and point M . Find all lines in space passing through M at distance d .

Tour 10.2

Grades 7 – 8

10.2.7-8.1. Twenty cubes of the same size and appearance are made of either aluminum or of heavier duralumin. How can one find the number of duralumin cubes using not more than 11 weighings on a balance without weights? (We assume that all cubes can be made of aluminum, but not all of duralumin.)

10.2.7-8.2. How many digits are there in the decimal expression of 2^{100} ?

10.2.7-8.3. Given 5 points on a plane, no three of which lie on one line. Prove that four of these points can be taken as vertices of a convex quadrilateral.

10.2.7-8.4. Prove that no convex 13-gon can be cut into parallelograms.

10.2.7-8.5. 101 numbers are selected from the set $1, 2, \dots, 200$. Prove that among the numbers selected there is a pair in which one number is divisible by the other.

Grades 9 – 10

10.2.9-10.1. In space, n wire triangles are situated so that any two of them have a common vertex and each vertex is the vertex of k triangles. Find all n and k for which this is possible.

10.2.9-10.2. In the numerical triangle

```

1
1 1 1
1 2 3 2 1
1 3 6 7 6 3 1
.....
    
```

each number is equal to the sum of the three nearest to it numbers from the row above it; if the number is at the beginning or at the end of a row then it is equal to the sum of its two nearest numbers or just to the nearest number above it (the lacking numbers above the given one are assumed to be zeros). Prove that each row, starting with the third one, contains an even number.

10.2.9-10.3. Inside a square, consider a convex quadrilateral and inside the quadrilateral, take a point A . It so happens that no three of the 9 points — the vertices of the square, of the quadrilateral and A — lie on one line. Prove that 5 of these points are vertices of a convex pentagon.

10.2.9-10.4. One number less than 16, and 99 other numbers are selected from the set $1, 2, \dots, 200$. Prove that among the selected numbers there are two such that one divides the other.

10.2.9-10.5. Prove that if the four faces of a tetrahedron are of the same area they are equal.

Olympiad 11 (1948)

Tour 11.1

Grades 7 – 8

11.1.7-8.1. The sum of the reciprocals of three positive integers is equal to 1. What are all the possible such triples?

11.1.7-8.2. Find all possible arrangements of 4 points on a plane, so that the distance between each pair of points is equal to either a or b . For what ratios of $a : b$ are such arrangements possible?

11.1.7-8.3. On a plane, n straight lines are drawn. Two domains are called *adjacent* if they border by a line segment. Prove that the domains into which the plane is divided by these lines can be painted two colors so that no two adjacent domains are of the same color.

Grades 9 – 10

11.1.9-10.1. Prove that if $\frac{2^n - 2}{n}$ is an integer, then so is $\frac{2^{2^n - 1} - 2}{2^n - 1}$.

11.1.9-10.2. Without tables and such (like calculators, virtually nonexistent in 1948) prove that

$$\frac{1}{\log_2 \pi} + \frac{1}{\log_5 \pi} > 2.$$

11.1.9-10.3. Consider two triangular pyramids $ABCD$ and $A'BCD$, with a common base BCD , and such that A' is inside $ABCD$. Prove that the sum of planar angles at vertex A' of pyramid $A'BCD$ is greater than the sum of planar angles at vertex A of pyramid $ABCD$.

11.1.9-10.4. Consider a circle and a point A outside it. We start moving from A along a closed broken line consisting of segments of tangents to the circle (the segment itself should not necessarily be tangent to the circle) and terminate back at A , as on Fig. 8. (On Fig. 8 the links of the broken line are solid.) We label parts of the segments with a plus sign if we approach the circle and with a minus sign otherwise. Prove that the sum of the lengths of the segments of our path, with the signs given, is zero.

FIGURE 8. (Probl. 11.1.9-10.4)

FIGURE 9. (Probl. 11.2.7-8.2)

Tour 11.2

Grades 7 – 8

11.2.7-8.1. Find all positive integer solutions of the equation

$$x^y = y^x \quad (x \neq y).$$

11.2.7-8.2. Let R and r be the radii of the circles circumscribed and inscribed, respectively, in a triangle. Prove that $R \geq 2r$, and that $R = 2r$ only for an equilateral triangle. (See Fig. 9.)

11.2.7-8.3. Can a figure have a greater than 1 and finite number of centers of symmetry?

11.2.7-8.4. The distance between the midpoints of the opposite sides of a convex quadrilateral is equal to a half sum of lengths of the other two sides. Prove that the first pair of sides is parallel.

11.2.7-8.5. Two legs of an angle α on a plane are mirrors. Prove that after several reflections in the mirrors any ray leaves in the direction opposite the one from which it came if and only if $\alpha = \frac{90^\circ}{n}$ for an integer n . Find the number of reflections.

Grades 9 – 10

11.2.9-10.1. Find all positive rational solutions of the equation

$$x^y = y^x \quad (x \neq y).$$

11.2.9-10.2*. What is the radius of the largest possible circle inscribed into a cube with side a ?

11.2.9-10.3. How many different integer solutions to the inequality $|x| + |y| < 100$ are there?

11.2.9-10.4. What is the greatest number of rays in space beginning at one point and forming pairwise obtuse angles?

11.2.9-10.5. Given three planar mirrors in space forming an octant (trihedral angle with right planar angles), prove that any ray of light coming into this mirrored octant leaves it, after several reflections in the mirrors, in the direction opposite to the one from which it came. Find the number of reflections. (Cf. Problem 11.2.7-8.5.)

Olympiad 12 (1949)

Tour 12.1

Grades 7 – 8

12.1.7-8.1. Prove that $27 \cdot 195^8 - 10 \cdot 887^8 + 10 \cdot 152^8$ is divisible by 26 460.

12.1.7-8.2. Prove that if a planar polygon has several axes of symmetry, then all of them intersect at one point.

12.1.7-8.3. Prove that $x^2 + y^2 + z^2 = 2xyz$ for integer x, y, z only if $x = y = z = 0$.

12.1.7-8.4. Consider a closed broken line of perimeter 1 on a plane. Prove that a disc of radius $\frac{1}{4}$ can cover this line.

12.1.7-8.5. Prove that for any triangle the circumscribed circle divides the line segment connecting the center of its inscribed circle with the center of one of the escribed circles in halves.

Grades 9 – 10

12.1.9-10.1. Find integers x, y, z, u such that

$$x^2 + y^2 + z^2 + u^2 = 2xyz u.$$

12.1.9-10.2. A finite solid body is symmetric about two distinct axes. Describe the position of the symmetry planes of the body.

12.1.9-10.3. Find the real roots of the equation

$$x^2 + 2ax + \frac{1}{16} = -a + \sqrt{a^2 + x - \frac{1}{16}} \quad (0 < a < \frac{1}{4}).$$

12.1.9-10.4. Given a set of $4n$ positive numbers such that any distinct choice of ordered foursomes of these numbers constitutes a geometric progression. Prove that at least 4 numbers of the set are identical.

12.1.9-10.5. Prove that if opposite sides of a hexagon are parallel and the diagonals connecting opposite vertices have equal lengths, a circle can be circumscribed around the hexagon.

Tour 12.2

Grades 7 – 8

12.2.7-8.1. There are 12 points on a circle. Four checkers, one red, one yellow, one green and one blue sit at neighboring points. In one move any checker can be moved four points to the left or right, onto the fifth point, if it is empty. If after several moves the checkers appear again at the four original points, how might their order have changed?

12.2.7-8.2. Consider two triangles, ABC and DEF , and any point O . We take any point X in $\triangle ABC$ and any point Y in $\triangle DEF$ and draw a parallelogram $OXYZ$. See Fig. 10. Prove that the locus of all possible points Z form a polygon. How many sides can it have? Prove that its perimeter is equal to the sum of perimeters of the original triangles.

12.2.7-8.3. Consider 13 weights of integer mass (in grams). It is known that any 6 of them may be placed onto two pans of a balance achieving equilibrium. Prove that all the weights are of equal mass.

12.2.7-8.4. The midpoints of alternative sides of a hexagon are connected by line segments. Prove that the intersection points of the medians of the two triangles obtained coincide.

FIGURE 10. (Probl. 12.2.7-8.2)

FIGURE

11. (Probl. 12.2.9-10.2)

12.2.7-8.5. Prove that some (or one) of any 100 integers can always be chosen so that the sum of the chosen integers is divisible by 100.

Grades 9 – 10

12.2.9-10.1. See Problem 12.2.7-8.1.

12.2.9-10.2. Construct a convex polyhedron of equal “bricks” shown in Fig. 11.

12.2.9-10.3. What is a centrally symmetric polygon of greatest area one can inscribe in a given triangle?

12.2.9-10.4*. Prove that a number of the form 2^n for a positive integer n may begin with any given combination of digits.

12.2.9-10.5. Two squares are said to be *juxtaposed* if their intersection is a point or a segment. Prove that it is impossible to juxtapose to a square more than eight non-overlapping squares of the same size.

Olympiad 13 (1950)

Tour 13.1

Grades 7 – 8

13.1.7-8.1. On a chess board, the boundaries of the squares are assumed to be black. Draw a circle of the greatest possible radius lying entirely on the black squares.

13.1.7-8.2. Given 555 weights: of 1 g, 2 g, 3 g, \dots , 555 g, divide them into three piles of equal mass.

13.1.7-8.3. See Problem 13.1.9-10.5 below for $n = 3$ circles.

13.1.7-8.4. Let a, b, c be the lengths of the sides of a triangle and A, B, C , the opposite angles. Prove that

$$Aa + Bb + Cc > \frac{Ab + Ac + Ba + Bc + Ca + Cb}{2}.$$

13.1.7-8.5. In a country, one can get from some point A to any other point either by walking, or by calling a cab, waiting for it, and then being driven. Every citizen always chooses the method of transportation that requires the least time. It turns out that the distances and the traveling times are as follows: 1 km takes 10 min; 2 km takes 15 min; 3 km takes 17.5 min. We assume that the speeds of the pedestrian and the cab, and the time spent waiting for cabs, are all constants. How long does it take to reach a point which is 6 km from A ?

Grades 9 – 10

13.1.9-10.1. Let A be an arbitrary angle; let B and C be acute angles. Is there an angle x such that

$$\sin x = \frac{\sin B \cdot \sin C}{1 - \cos B \cdot \cos C \cdot \cos A}?$$

13.1.9-10.2. Two triangular pyramids have common base. One pyramid contains the other. Can the sum of the lengths of the edges of the inner pyramid be longer than that of the outer one?

13.1.9-10.3. Arrange 81 weights of $1^2, 2^2, \dots, 81^2$ (all in grams) into three piles of equal mass.

13.1.9-10.4. Solve the equation

$$\sqrt{x+3-4\sqrt{x-1}} + \sqrt{x+8-6\sqrt{x-1}} = 1.$$

13.1.9-10.5. We are given n circles O_1, O_2, \dots, O_n , passing through one point O . Let A_1, \dots, A_n denote the second intersection points of O_1 with O_2, O_2 with O_3 , etc., O_n with O_1 , respectively. We choose an arbitrary point B_1 on O_1 and draw a line segment through A_1 and B_1 to the second intersection with O_2 at B_2 , then draw a line segment through A_2 and B_2 to the second intersection with O_3 at B_3 , etc., until we get a point B_n on O_n . We draw the line segment through B_n and A_n to the second intersection with O_1 at B_{n+1} . If B_k and A_k coincide for some k , we draw the tangent to O_k through A_k until this tangent intersects O_{k+1} at B_{k+1} . Prove that B_{n+1} coincides with B_1 .

Tour 13.2

Grades 7 – 8

13.2.7-8.1. In a convex 13-gon all diagonals are drawn, dividing it into smaller polygons. What is the greatest number of sides can these polygons have? (Cf. Problem 13.2.9-10.1.)

13.2.7-8.2. Prove that

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdots \frac{99}{100} < \frac{1}{10}.$$

13.2.7-8.3. A circle is inscribed in a triangle and a square is circumscribed around this circle so that no side of the square is parallel to any side of the triangle. Prove that less than half of the square's perimeter lies outside the triangle.

13.2.7-8.4*. On a circle, 20 points are chosen. Ten non-intersecting chords without mutual endpoints connect some of the points chosen. How many distinct such arrangements are there?

Grades 9 – 10

13.2.9-10.1. In a convex 1950-gon all diagonals are drawn, dividing it into smaller polygons. What is the greatest number of sides can these polygons have? (Cf. Problem 13.2.7-8.1.)

13.2.9-10.2. The numbers 1, 2, 3, \dots , 101 are written in a row in some order. Prove that it is always possible to erase 90 of the numbers so that the remaining 11 numbers remain arranged in either increasing or decreasing order.

13.2.9-10.3. A spatial quadrilateral is circumscribed around a sphere. Prove that all the tangent points lie in one plane.

13.2.9-10.4. Is it possible to draw 10 bus routes with stops such that for any 8 routes there is a stop that does not belong to any of the routes, but any 9 routes pass through all the stops?

Olympiad 14 (1951)

Tour 14.1

Grades 7 – 8

14.1.7-8.1. Prove that $x^{12} - x^9 + x^4 - x + 1 > 0$ for all x .

14.1.7-8.2. Let $ABCD$ and $A'B'C'D'$ be two convex quadrilaterals whose corresponding sides are equal, i.e., $AB = A'B', BC = B'C'$, etc. Prove that if $\angle A > \angle A'$, then $\angle B < \angle B', \angle C > \angle C', \angle D < \angle D'$.

14.1.7-8.3. Which number is greater:

$$\frac{2.00\,000\,000\,004}{(1.00\,000\,000\,004)^2 + 2.00\,000\,000\,004} \quad \text{or} \quad \frac{2.00\,000\,000\,002}{(1.00\,000\,000\,002)^2 + 2.00\,000\,000\,002}?$$

14.1.7-8.4. Given an isosceles trapezoid $ABCD$ and a point P . Prove that a quadrilateral can be constructed from segments PA, PB, PC, PD .

14.1.7-8.5. Given a chain of 60 links each weighing 1 g. Find the smallest number of links that need to be broken if we want to be able to get from the obtained parts all weights 1 g, 2 g, \dots , 59 g, 60 g? A broken link also weighs 1 g. (Cf. Problem 14.1.9-10.4.)

Grades 9 – 10

14.1.9-10.1. Find the first three figures after the decimal point in the decimal expression of the number

$$\frac{0.123456789101112\dots495051}{0.515049\dots121110987654321}$$

14.1.9-10.2. One side of a convex polygon is equal to a , the sum of exterior angles at the vertices not adjacent to this side are equal to 120° . Among such polygons, find the polygon of the largest area.

14.1.9-10.3. We have two concentric circles. A polygon is circumscribed around the smaller circle and is contained entirely inside the greater circle. Perpendiculars from the common center of the circles to the sides of the polygon are extended till they intersect the greater circle. Each of the points obtained is connected with the endpoints of the corresponding side of the polygon (Fig. 12). When is the resulting star-shaped polygon the unfolding of a pyramid?

FIGURE 12. (Probl. 14.1.9-10.3)

FIGURE

13. (Probl. 14.1.9-10.5)

14.1.9-10.4. Given a chain of 150 links each weighing 1 g. Find the smallest number of links that need to be broken if we want to be able to get from the obtained parts all weights 1 g, 2 g, \dots , 149 g, 150 g? A broken link also weighs 1 g. (Cf. Problem 14.1.7-8.5.)

14.1.9-10.5. Given three equidistant parallel lines. Express by points of the corresponding lines the values of the resistance, voltage and current in a conductor so as to obtain the voltage $V = I \cdot R$ by connecting with a ruler the points denoting the resistance R and the current I . (Each point of each scale denotes only one number). See Fig. 13.

Tour 14.2

Grades 7 – 8

14.2.7-8.1. Prove that the number $1\underbrace{00\dots00}_{49 \text{ zeroes}}5\underbrace{00\dots00}_{99 \text{ zeroes}}1$ is not the cube of any integer.

14.2.7-8.2*. On a plane, given points A, B, C and angles $\angle D, \angle E, \angle F$ each less than 180° and the sum equal to 360° , construct with the help of ruler and protractor a point O such that $\angle AOB = \angle D$, $\angle BOC = \angle E$ and $\angle COA = \angle F$.

14.2.7-8.3. Prove that the sum $1^3 + 2^3 + \dots + n^3$ is a perfect square for all n .

14.2.7-8.4. What figure can the central projection of a triangle be? (The center of the projection does not lie on the plane of the triangle.)

14.2.7-8.5. To prepare for an Olympiad 20 students went to a coach. The coach gave them 20 problems and it turned out that (a) each of the students solved two problems and (b) each problem was solved by two students. Prove that it is possible to organize the coaching so that each student would discuss one of the problems that (s)he had solved, and so that all problems would be discussed.

14.2.7-8.6. Dividing $x^{1951} - 1$ by $P(x) = x^4 + x^3 + 2x^2 + x + 1$ one gets a quotient and a remainder. Find the coefficient of x^{14} in the quotient.

Grades 9 – 10

14.2.9-10.1. A sphere is inscribed in an n -angled pyramid. Prove that if we align all side faces of the pyramid with the base plane, flipping them around the corresponding edges of the base, then (1) all tangent points of these faces to the sphere would coincide with one point, H , and (2) the vertices of the faces would lie on a circle centered at H .

14.2.9-10.2*. Given several numbers each of which is less than 1951 and the least common multiple of any two of which is greater than 1951. Prove that the sum of their reciprocals is less than 2.

14.2.9-10.3. Among all orthogonal projections of a regular tetrahedron to all possible planes, find the projection of the greatest area.

14.2.9-10.4. Consider a curve with the following property: *inside the curve one can move an inscribed equilateral triangle so that each vertex of the triangle moves along the curve and draws the whole curve.* Clearly, every circle possesses the property. Find a closed planar curve without self-intersections, that has the property but is not a circle.

14.2.9-10.5*. A bus route has 14 stops (counting the first and the last). A bus cannot carry more than 25 passengers. We assume that a passenger *takes a bus from A to B* if (s)he enters the bus at A and gets off at B. Prove that for any bus route

a) there are 8 distinct stops $A_1, B_1, A_2, B_2, A_3, B_3, A_4, B_4$ such that

$$\text{no passenger rides from } A_k \text{ to } B_k \text{ for all } k = 1, 2, 3, 4; \quad (*)$$

b) there might not exist 10 distinct stops $A_1, B_1, \dots, A_5, B_5$ with property (*).

Olympiad 15 (1952)

Tour 15.1

Grade 7

15.1.7.1. The circle is inscribed in $\triangle ABC$. Let L, M, N be the tangent points of the circle with sides AB, AC, BC , respectively. Prove that $\angle MLN$ is always an acute angle.

15.1.7.2. Prove the identity:

$$(ax + by + cz)^2 + (bx + cy + az)^2 + (cx + ay + bz)^2 = (cx + by + az)^2 + (bx + ay + cz)^2 + (ax + cy + bz)^2.$$

15.1.7.3. Prove that if all faces of a parallelepiped are equal parallelograms, they are rhombuses.

15.1.7.4. See Problem 15.1.8.2 below. When should the girl C leave N for A and B to arrive simultaneously in N ?

Grade 8

15.1.8.1. Prove that if the orthocenter divides all heights of a triangle in the same proportion, the triangle is equilateral.

15.1.8.2. Two men, A and B , set out from town M to town N , which is 15 km away. Their walking speed is 6 km/hr. They also have a bicycle which they can ride at 15 km/hr. Both A and B start simultaneously, A walking and B riding a bicycle until B meets a pedestrian girl, C , going from N to M . Then B lends his bicycle to C and proceeds on foot; C rides the bicycle until she meets A and gives A the bicycle which A rides until he reaches N . The speed of C is the same as that of A and B . The time spent by A and B on their trip is measured from the moment they started from M until the arrival of the last of them at N . When should C leave N to minimize this time?

15.1.8.3. Prove the identity:

$$\begin{aligned} &(ax + by + cz + du)^2 + (bx + cy + dz + au)^2 + \\ &\quad (cx + dy + az + bu)^2 + (dx + ay + bz + cu)^2 = \\ &(dx + cy + bz + au)^2 + (cx + by + az + du)^2 + \\ &\quad (bx + ay + dz + cu)^2 + (ax + dy + cz + bu)^2. \end{aligned}$$

15.1.8.4. See Problem 15.1.7.3.

Grade 9

15.1.9.1. Given a geometric progression whose denominator q is an integer not equal to 0 or -1 , prove that the sum of two or more terms in this progression cannot equal any other term in it.

15.1.9.2. Prove that if $|x| < 1$ and $|y| < 1$, then $\left| \frac{x-y}{1-xy} \right| < 1$.

15.1.9.3. $\triangle ABC$ is divided by a straight line BD into two triangles. Prove that the sum of the radii of circles inscribed in triangles ABD and DBC is greater than the radius of the circle inscribed in $\triangle ABC$. (See Fig. 14.)

FIGURE 14. (Probl. 15.1.9.3)

FIGURE 15. (Probl. 15.2.8.2)

15.1.9.4. A sequence of integers is constructed as follows: a_1 is an arbitrary three-digit number, a_2 is the sum of squares of the digits of a_1 , a_3 is the sum of squares of the digits of a_2 , etc. Prove that either 1 or 4 must occur in the sequence a_1, a_2, a_3, \dots

15.1.9.5. See Problem 15.1.10.5 below.

Grade 10

15.1.10.1. How $\arcsin(\cos(\arcsin x))$ and $\arccos(\sin(\arccos x))$ are related with each other?

15.1.10.2. Prove that $(1 - x)^n + (1 + x)^n < 2^n$ for an integer $n \geq 2$ and $|x| < 1$.

15.1.10.3. A sphere with center at O is inscribed in a trihedral angle with vertex S . Prove that the plane passing through the three tangent points is perpendicular to OS .

15.1.10.4. Prove that if for any positive p all roots of the equation

$$ax^2 + bx + c + p = 0$$

are real and positive then $a = 0$.

15.1.10.5. Given three skew lines. Prove that they are pair-wise perpendicular to their pair-wise perpendiculars.

Tour 15.2

Grade 7

15.2.7.1. Solve the system of equations

$$\begin{cases} 1 - x_1x_2 = 0, \\ 1 - x_2x_3 = 0, \\ \dots\dots\dots \\ 1 - x_{14}x_{15} = 0, \\ 1 - x_{15}x_1 = 0. \end{cases}$$

(Cf. Problem 15.2.9.1 below.)

15.2.7.2. In a convex quadrilateral $ABCD$, let $AB + CD = BC + AD$. Prove that the circle inscribed in $\triangle ABC$ is tangent to the circle inscribed in $\triangle ACD$.

15.2.7.3. Prove that if the square of a number begins with 0.9...9 (100 nines), then the number itself begins with 0.9...9 (not less than 100 nines). (Cf. Problem 15.2.8.1 below).

15.2.7.4. Given a line segment AB , find the set of vertices C that form an acute triangle ABC .

Grade 8

15.2.8.1. Calculate $\sqrt{0.9\dots9}$ (60 nines) to 60 decimal places.

15.2.8.2. From a point C , tangents CA and CB are drawn to a circle O . From an arbitrary point N on the circle, perpendiculars ND , NE , NF are dropped to AB , CA and CB , respectively. Prove that the length of ND is the mean proportional of the lengths of NE and NF . (See Fig. 15).

15.2.8.3. Seven chips are numbered 1, 2, 3, 4, 5, 6, 7. Prove that none of the seven-digit numbers formed by these chips is divisible by any other of these seven-digit numbers.

15.2.8.4. 99 straight lines divide a plane into n parts. Find all possible values of n less than 199.

Grade 9

15.2.9.1. Solve the system of equations

$$\left\{ \begin{array}{l} 1 - x_1x_2 = 0, \\ 1 - x_2x_3 = 0, \\ \dots\dots\dots \\ 1 - x_{n-1}x_n = 0, \\ 1 - x_nx_1 = 0. \end{array} \right.$$

How does the solution vary for distinct values of n ?

15.2.9.2. How to arrange three right circular cylinders of diameter $\frac{a}{2}$ and height a into an empty cube with side a so that the cylinders could not change position inside the cube? Each cylinder can, however, rotate about its axis of symmetry.

15.2.9.3. See Problem 15.2.8.3.

15.2.9.4. In an isosceles triangle $\triangle ABC$, $\angle ABC = 20^\circ$ and $BC = AB$. Points P and Q are chosen on sides BC and AB , respectively, so that $\angle PAC = 50^\circ$ and $\angle QCA = 60^\circ$. Prove that $\angle PQC = 30^\circ$. (See Fig. 16).

FIGURE 16. (Probl. 15.2.9.4)

FIGURE 17. (Probl. 16.1.8.1)

15.2.9.5. 200 soldiers occupy in a rectangle (military call it a *square* and educated military a *carrée*): 20 men (per row) times 10 men (per column).

In each row, we consider the tallest man (if some are of equal height, choose any of them) and of the 10 men considered we select the shortest (if some are of equal height, choose any of them). Call him A .

Next the soldiers assume their initial positions and in each column the shortest soldier is selected; of these 20, the tallest is chosen. Call him B .

Two colonels bet on which of the two soldiers chosen by these two distinct procedures is taller: A or B . Which colonel wins the bet?

Grade 10

15.2.10.1. Prove that for arbitrary fixed a_1, a_2, \dots, a_{31} the sum

$$\cos 32x + a_{31} \cos 31x + \dots + a_2 \cos 2x + a_1 \cos x$$

can take both positive and negative values as x varies.

15.2.10.2. See Problem 15.2.9.2.

15.2.10.3. Prove that for any integer a the polynomial $3x^{2n} + ax^n + 2$ cannot be divided by $2x^{2m} + ax^m + 3$ without a remainder.

15.2.10.4. See Problem 15.2.9.4.

15.2.10.5. See Problem 15.2.9.5.

Olympiad 16 (1953)

Tour 16.1

Grade 7

16.1.7.1. Prove that the sum of angles at the longer base of a trapezoid is less than the sum of angles at the shorter base.

16.1.7.2. Find the smallest number of the form $1\dots 1$ in its decimal expression which is divisible by $3\dots 3$ (100 three's).

16.1.7.3. Divide a segment in halves using a right triangle. (With a right triangle one can draw straight lines and erect perpendiculars but cannot drop perpendiculars.)

16.1.7.4. Prove that $n^2 + 8n + 15$ is not divisible by $n + 4$ for any positive integer n .

Grade 8

16.1.8.1. Three circles are pair-wise tangent to each other. Prove that the circle passing through the three tangent points is perpendicular to each of the initial three circles; see Fig. 17.

16.1.8.2. Prove that if in the following fraction we have n radicals in the numerator and $n - 1$ in the denominator, then

$$\frac{2 - \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}{2 - \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}} > \frac{1}{4}.$$

16.1.8.3. See Problem 16.1.7.2.

16.1.8.4. See Problem 16.1.7.3.

Grade 9

16.1.9.1. On the plane find the locus of points whose coordinates satisfy $\sin(x + y) = 0$.

16.1.9.2. Let AB and A_1B_1 be two skew segments, O and O_1 their respective midpoints. Prove that OO_1 is shorter than a half sum of AA_1 and BB_1 .

16.1.9.3. Prove that the polynomial $x^{200} \cdot y^{200} + 1$ cannot be represented in the form $f(x) \cdot g(y)$, where f and g are polynomials of only x and y , respectively.

16.1.9.4. Let A be a vertex of a regular star-shaped pentagon, the angle at A being less than 180° and the broken line $AA_1BB_1CC_1DD_1EE_1$ being its contour. Lines AB and DE meet at F . Prove that polygon $ABB_1CC_1DED_1$ has the same area as the quadrilateral AD_1EF .

16.1.9.5. See Problem 16.1.8.2

Grade 10

16.1.10.1. See Problem 16.1.9.1.

16.1.10.2. Given a right circular cone and a point A . Find the set of vertices of cones equal to the given one, with axes parallel to that of the given one, and with A inside them.

16.1.10.3. See Problem 16.1.9.3.

16.1.10.4. See Problem 16.1.9.4.

16.1.10.5. See Problem 16.1.8.2.

Tour 16.2

Grade 7

16.2.7.1. Prove that $GCD(a + b, LCM(a, b)) = GCD(a, b)$ for any a, b .

16.2.7.2. A quadrilateral is circumscribed around a circle. Its diagonals intersect at the center of the circle. Prove that the quadrilateral is a rhombus.

16.2.7.3. On a plane, 11 gears are arranged so that the teeth of the first gear mesh with the teeth of the second gear, the teeth of the second gear with those of the third gear, etc., and the teeth of the last gear mesh with those of the first gear. Can the gears rotate? (See Problem 16.2.8.4 below.)

16.2.7.4. Inside a convex 1000-gon, 500 points are selected so that no three of the 1500 points — the ones selected and the vertices of the polygon — lie on the same straight line. This 1000-gon is then divided into triangles so that all 1500 points are vertices of the triangles, and so that these triangles have no other vertices. How many triangles will there be?

16.2.9.3. Given the equations

$$(1) \quad ax^2 + bx + c = 0$$

$$(2) \quad -ax^2 + bx + c = 0$$

prove that if x_1 and x_2 are some roots of equations (1) and (2), respectively, then there is a root x_3 of the equation $\frac{a}{2}x^2 + bx + c = 0$ such that either $x_1 \leq x_3 \leq x_2$ or $x_1 \geq x_3 \geq x_2$.

16.2.9.4. Given a 101×200 sheet of graph paper, we start moving from a corner square in the direction of the square's diagonal (not the sheet's diagonal) to the border of the sheet, then change direction obeying the laws of light's reflection. Will we ever reach a corner square?

16.2.9.5. Divide a cube into three equal pyramids.

Grade 10

16.2.10.1. Find roots of the equation

$$1 - \frac{x}{1} + \frac{x(x-1)}{1 \cdot 2} - \dots + \frac{(-1)^n x(x-1) \dots (x-n+1)}{n!} = 0.$$

16.2.10.2. See Problem 16.2.9.2.

16.2.10.3. Let $x_0 = 10^9$, $x_n = \frac{x_{n-1}^2 + 2}{2x_{n-1}}$ for $n > 0$. Prove that $0 < x_{36} - \sqrt{2} < 10^{-9}$.

16.2.10.4. See Problem 16.2.9.5.

16.2.10.5. A knight stands on an infinite chess board. Find all places it can reach in exactly $2n$ moves.

Olympiad 17 (1954)

Tour 17.1

Grade 7

17.1.7.1. A regular star-shaped hexagon is split into 4 parts. Construct from them a convex polygon.

17.1.7.2. Given two convex polygons, $A_1A_2 \dots A_n$ and $B_1B_2 \dots B_n$ such that $A_1A_2 = B_1B_2$, $A_2A_3 = B_2B_3$, \dots , $A_nA_1 = B_nB_1$ and $n - 3$ angles of one polygon are equal to the respective angles of the other. Find whether these polygons are equal.

17.1.7.3. Find a four-digit number whose division by two given distinct one-digit numbers goes along the following lines:

FIGURE 19. (Probl. 17.1.7.3)

17.1.7.4. Are there integers m and n such that $m^2 + 1954 = n^2$?

17.1.7.5. Define the maximal value of the ratio of a three-digit number to the sum of its digits.

Grade 8

17.1.8.1*. Cut out of a 3×3 square an unfolding of the cube with edge 1.

17.1.8.2. From an arbitrary point O inside a convex n -gon, perpendiculars are dropped to the (extensions of the) sides of the n -gon. Along each perpendicular a vector is constructed, starting from O , directed towards the side onto which the perpendicular is dropped, and of length equal to half the length of the corresponding side; see Fig. 20. Find the sum of these vectors.

17.1.8.3. See Problem 17.1.7.3.

17.1.8.4. Find all solutions of the system consisting of 3 equations:

$$x \left(1 - \frac{1}{2n}\right) + y \left(1 - \frac{1}{2n+1}\right) + z \left(1 - \frac{1}{2n+2}\right) = 0 \text{ for } n = 1, 2, 3.$$

Tour 17.2

Grade 7

17.2.7.1. Given a piece of graph paper with a letter assigned to each vertex of every square such that on every segment connecting two vertices that have the same letter and are on the same line of the mesh, there is at least one vertex with another letter. What is the least number of distinct letters needed to plot such a picture?

17.2.7.2*. Solve the system

$$\begin{cases} 10x_1 + 3x_2 + 4x_3 + x_4 + x_5 = 0, \\ 11x_2 + 2x_3 + 2x_4 + 3x_5 + x_6 = 0, \\ 15x_3 + 4x_4 + 5x_5 + 4x_6 + x_7 = 0, \\ 2x_1 + x_2 - 3x_3 + 12x_4 - 3x_5 + x_6 + x_7 = 0, \\ 6x_1 - 5x_2 + 3x_3 - x_4 + 17x_5 + x_6 = 0, \\ 3x_1 + 2x_2 - 3x_3 + 4x_4 + x_5 - 16x_6 + 2x_7 = 0, \\ 4x_1 - 8x_2 + x_3 + x_4 + 3x_5 + 19x_7 = 0. \end{cases}$$

17.2.7.3. How many axes of symmetry can a heptagon have?

17.2.7.4. Let 1, 2, 3, 5, 6, 7, 10, \dots , N be all the divisors of

$$N = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31$$

(the product of primes 2 to 31) written in increasing order. Below this series of divisors, write the following series of 1's or -1 's: write 1 below any number that factors into an even number of prime factors and below a -1 ; write -1 below the remaining numbers. Prove that the sum of the series of 1's and -1 's is equal to 0. (Cf. Problem 17.2.8.5.)

17.2.7.5. The map of a town shows a plane divided into equal equilateral triangles. The sides of these triangles are streets and their vertices are intersections; 6 streets meet at each junction. Two cars start simultaneously in the same direction and at the same speed from points A and B situated on the same street (the same side of a triangle). After any intersection an admissible route for each car is either to proceed in its initial direction or turn through 120° to the right or to the left; see Fig. 21. Can these cars meet? (Either prove that these cars won't meet or describe a route by which they will meet.)

FIGURE 21. (Probl. 17.2.7.5)

Grade 8

17.2.8.1. A 17×17 square is cut out of a sheet of graph paper. Each cell of this square has one of the numbers from 1 to 70. Prove that there are 4 distinct squares whose centers A , B , C , D are the vertices of a parallelogram such that $AB \parallel CD$, moreover, the sum of the numbers in the squares with centers A and C is equal to that in the squares with centers B and D .

17.2.8.2. Given four straight lines, m_1 , m_2 , m_3 , m_4 , intersecting at O and numbered clockwise with O as the center of the clock, we draw a line through an arbitrary point A_1 on m_1 parallel to m_4 until the line meets m_2 at A_2 . We draw a line through A_2 parallel to m_1 until it meets m_3 at A_3 . We also draw a line through A_3 parallel to m_2 until it meets m_4 at A_4 . Now, we draw a line through A_4 parallel to m_3 until it meets m_1 at B . Prove that $OB \leq \frac{OA_1}{2}$. (See Fig. 22.)

17.2.8.3. See Problem 17.2.7.2.

FIGURE 22. (Probl. 17.2.8.2)

17.2.8.4. See Problem 17.2.7.3.

17.2.8.5. Let $1, 2, 3, 5, 6, 7, 10, \dots, N$ be all the divisors of

$$N = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37$$

(the product of primes 2 to 37) written in increasing order. Below this series of divisors, write the following series of 1's or -1 's: write 1 below any number that factors into an even number of prime factors and below a -1 ; write -1 below the remaining numbers. Prove that the sum of the series of 1's and -1 's is equal to 0. (Cf. Problem 17.2.7.4.)

Grade 9

17.2.9.1. Rays l_1 and l_2 pass through a point O . Segments OA_1 and OB_1 on l_1 , and OA_2 and OB_2 on l_2 , are drawn so that $\frac{OA_1}{OA_2} \neq \frac{OB_1}{OB_2}$. Find the set of all intersection points of lines A_1A_2 and B_1B_2 as l_2 rotates around O while l_1 is fixed.

17.2.9.2. See Problem 17.2.8.2; prove that $OB \leq \frac{1}{4}OA_1$. (See Fig. 22.)

17.2.9.3*. Positive numbers x_1, x_2, \dots, x_{100} satisfy the system

$$\begin{cases} x_1^2 + x_2^2 + \dots + x_{100}^2 > 10\,000, \\ x_1 + x_2 + \dots + x_{100} < 300. \end{cases}$$

Prove that among these numbers there are three whose sum is greater than 100.

17.2.9.4. Given a sequence of numbers a_1, a_2, \dots, a_{15} , one can always construct a new sequence b_1, b_2, \dots, b_{15} , where b_i is equal to the number of terms in the sequence $\{a_k\}_{k=1}^{15}$ less than a_i ($i = 1, 2, \dots, 15$). Is there a sequence $\{a_k\}_{k=1}^{15}$ for which the sequence $\{b_k\}_{k=1}^{15}$ is $1, 0, 3, 6, 9, 4, 7, 2, 5, 8, 8, 5, 10, 13, 13$?

17.2.9.5. Consider five segments $AB_1, AB_2, AB_3, AB_4, AB_5$. From each point B_i there can exit either 5 segments or no segments at all, so that the endpoints of any two segments of the resulting graph (system of segments) do not coincide. (See Fig. 23.) Can the number of free endpoints of the segments thus constructed be equal to 1001? (A *free endpoint* is an endpoint from which no segment begins.)

FIGURE 23. (Probl. 17.2.9.5)

Grade 10

17.2.10.1. How many planes of symmetry can a triangular pyramid have?

17.2.10.2. See Problem 17.2.9.2.

17.2.10.3. See Problem 17.2.9.3.

17.2.10.4. The absolute values of all roots of the quadratic equation $x^2 + Ax + B = 0$ and $x^2 + Cx + D = 0$ are less than 1. Prove that so are absolute values of the roots of the quadratic equation

$$x^2 + \frac{A+C}{2}x + \frac{B+D}{2} = 0.$$

17.2.10.5. Consider the set of all 10-digit numbers expressible with the help of figures 1 and 2 only. Divide it into two subsets so that the sum of any two numbers of the same subset is a number which is written with not less than two 3's.

Olympiad 18 (1955)

Tour 18.1

Grade 7

18.1.7.1. The numbers $1, 2, \dots, 49$ are arranged in a square table as follows:

1	2	...	7
8	9	...	14
...
43	44	...	49

Among these numbers we select an arbitrary number and delete from the table the row and the column which contain this number. We do the same with the remaining table of 36 numbers, etc., 7 times. Find the sum of the numbers selected. (See Problem 18.1.9.1 below.)

18.1.7.2. We are given a right triangle ABC and the median BD drawn from the vertex B of the right angle. Let the circle inscribed in $\triangle ABD$ be tangent to side AD at K . Find the angles of $\triangle ABC$ if K divides AD in halves.

18.1.7.3. Consider an equilateral triangle $\triangle ABC$ and points D and E on the sides AB and BC such that $AE = CD$. Find the locus of intersection points of AE with CD as points D and E vary.

18.1.7.4. Is there an integer n such that $n^2 + n + 1$ is divisible by 1955?

18.1.7.5. Find all rectangles that can be cut into 13 equal squares.

Grade 8

18.1.8.1. Let a, b, n be positive integers, $b < 10$ and $2^n = 10a + b$. Prove that if $n > 3$, then 6 divides ab .

18.1.8.2. Consider a quadrilateral $ABCD$ and points K, L, M, N on sides AB, BC, CD and AD , respectively, such that $KB = BL = a$, $MD = DN = b$ and $KL \nparallel MN$. Find the set of all the intersection points of KL with MN as a and b vary.

18.1.8.3. A square table with 49 small squares is filled with numbers 1 to 7 so that in each row and in each column all numbers from 1 to 7 are present. Let the table be symmetric through the main diagonal. Prove that on this diagonal all the numbers 1, 2, 3, ..., 7 are present. (See Problem 18.1.10.1 below.)

18.1.8.4. Which convex domains on a plane can contain an entire straight line?

18.1.8.5. There are four points A, B, C, D on a circle. Circles are drawn through each pair of neighboring points. Denote the intersection points of neighboring circles by A_1, B_1, C_1, D_1 . (Some of these points may coincide with previously given ones.) Prove that points A_1, B_1, C_1, D_1 lie on one circle; see Fig. 24.

Grade 9

18.1.9.1. The numbers $1, 2, \dots, k^2$ are arranged in a square table as follows:

1	2	...	k
$k+1$	$k+2$...	$2k$
...
$(k-1)k+1$	$(k-1)k+2$...	k^2

FIGURE 24. (Probl. 18.1.8.5)

Among these numbers we select an arbitrary number and delete from the table the row and the column which contain this number. We do the same with the remaining table of $(k-1)^2$ numbers, etc., k times. Find the sum of the numbers selected.

18.1.9.2. Given two distinct nonintersecting circles none of which is inside the other, see Fig. 25. Find the locus of the midpoints of all segments whose endpoints lie on the circles.

FIGURE 25. (Probl. 18.1.9.2)

18.1.9.3. Find all real solutions of the system:

$$\begin{cases} x^3 + y^3 = 1, \\ x^4 + y^4 = 1. \end{cases}$$

18.1.9.4. Suppose that primes a_1, a_2, \dots, a_p form an increasing arithmetic progression and $a_1 > p$. Prove that if p is a prime, then the difference of the progression is divisible by p .

18.1.9.5. Inside $\triangle ABC$, there is fixed a point D such that $AC - DA > 1$ and $BC - BD > 1$. Prove that $EC - ED > 1$ for any point E on segment AB ; see Fig. 26.

Grade 10

18.1.10.1. A square table with n^2 small squares is filled with numbers 1 to n so that in each row and in each column all numbers from 1 to n are present. Let n be odd and the table be symmetric through the main diagonal. Prove that on this diagonal all the numbers 1, 2, 3, \dots , n are present.

18.1.10.2. See Problem 18.1.9.3.

18.1.10.3. See Problem 18.1.9.5.

18.1.10.4. Given a trihedral angle with vertex O . Find whether there is a planar section ABC such that the angles $\angle OAB, \angle OBA, \angle OBC, \angle OCB, \angle OAC, \angle OCA$ are acute?

FIGURE 26. (Probl. 18.1.9.5)

Tour 18.2

Grade 7

18.2.7.1. Find integer solutions of the equation

$$x^3 - 2y^3 - 4z^3 = 0.$$

18.2.7.2. The quadratic expression $ax^2 + bx + c$ is the 4-th power (of an integer) for any integer x . Prove that $a = b = 0$.

18.2.7.3. The centers O_1 , O_2 and O_3 of circles escribed about $\triangle ABC$ are connected. Prove that $\triangle O_1O_2O_3$ is an acute-angled one.

18.2.7.4. 25 chess players are going to participate in a chess tournament. All are on distinct skill levels, and of the two players the one who plays better always wins. What is the least number of games needed to select the two best players?

18.2.7.5. Cut a rectangle into 18 rectangles so that no two adjacent ones form a rectangle.

Grade 8

18.2.8.1*. The quadratic expression $ax^2 + bx + c$ is a square (of an integer) for any integer x . Prove that $ax^2 + bx + c = (dx + e)^2$ for some integers d and e .

18.2.8.2*. Two circles are tangent to each other externally, and to a third one from the inside. Two common tangents to the first two circles are drawn, one outer and one inner. Prove that the inner tangent divides in halves the arc intercepted by the outer tangent on the third circle. (Cf. Problem 20.2.9.5.)

18.2.8.3. A point O inside a convex n -gon $A_1A_2 \dots A_n$ is connected with segments to its vertices. The sides of this n -gon are numbered 1 to n (distinct sides have distinct numbers). The segments OA_1, OA_2, \dots, OA_n are similarly numbered.

a) For $n = 9$ find a numeration such that the sum of the sides' numbers is the same for all triangles $A_1OA_2, A_2OA_3, \dots, A_nOA_1$.

b) Prove that for $n = 10$ there is no such numeration.

18.2.8.4. Let the inequality

$$Aa(Bb + Cc) + Bb(Aa + Cc) + Cc(Aa + Bb) > \frac{ABc^2 + BCa^2 + CAb^2}{2}$$

with given $a > 0, b > 0, c > 0$ hold for all $A > 0, B > 0, C > 0$. Is it possible to construct a triangle with sides of lengths a, b, c ?

18.2.8.5. Find all numbers a such that (1) for a fixed positive integer N all numbers $[a], [2a], \dots, [Na]$ are distinct and (2) all numbers $\left[\frac{1}{a}\right], \left[\frac{2}{a}\right], \dots, \left[\frac{N}{a}\right]$, are distinct.

Grade 9

18.2.9.1. Given $\triangle ABC$, points C_1, A_1, B_1 on sides AB, BC, CA , respectively, such that

$$\frac{AC_1}{C_1B} = \frac{BA_1}{A_1C} = \frac{CB_1}{B_1A} = \frac{1}{n}$$

and points C_2, A_2, B_2 on sides A_1B_1, B_1C_1, C_1A_1 of $\triangle A_1B_1C_1$, respectively, such that

$$\frac{A_1C_2}{C_2B_1} = \frac{B_1A_2}{A_2C_1} = \frac{C_1B_2}{B_2A_1} = n.$$

Prove that $A_2C_2 \parallel AC$, $C_2B_2 \parallel CB$, $B_2A_2 \parallel BA$.

18.2.9.2. On the numerical line, arrange a system of closed segments of length 1 without common points (endpoints included) so that any infinite arithmetic progression with any difference and any first term has a common point with a segment of the system.

18.2.9.3. Prove that the equation

$$x^n - a_1x^{n-1} - a_2x^{n-2} - \dots - a_{n-1}x - a_n = 0, \quad \text{where } a_1 \geq 0, a_2 \geq 0, \dots, a_n \geq 0,$$

cannot have two positive roots.

18.2.9.4. See Problem 18.2.8.2.

18.2.9.5. Five men play several sets of dominoes (two against two) so that each player has each other player once as a partner and two times as an opponent. Find the number of sets and all ways to arrange the players.

Grade 10

18.2.10.1. Prove that if $\frac{p}{q}$ is an irreducible rational number that serves as a root of the polynomial

$$f(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$$

with integer coefficients, then $p - kq$ is a divisor of $f(k)$ for any integer k .

18.2.10.2. See Problem 18.2.9.2.

18.2.10.3. A right circular cone stands on plane P . The radius of the cone's base is r , its height is h . A source of light is placed at distance H from the plane, and distance 1 from the axis of the cone. What is the illuminated part of the disc of radius R , that belongs to P and is concentric with the disc forming the base of the cone?

18.2.10.4. What greatest number of triples of points can be selected from 1955 given points, so that each two triples have one common point?

18.2.10.5. Consider $\triangle A_0B_0C_0$ and points C_1, A_1, B_1 on its sides A_0B_0, B_0C_0, C_0A_0 , points C_2, A_2, B_2 on the sides A_1B_1, B_1C_1, C_1A_1 of $\triangle A_1B_1C_1$, respectively, etc., so that

$$\frac{A_0B_1}{B_1C_0} = \frac{B_0C_1}{C_1A_0} = \frac{C_0A_1}{A_1B_0} = k, \quad \frac{A_1B_2}{B_2C_1} = \frac{B_1C_2}{C_2A_1} = \frac{C_1A_2}{A_2B_1} = \frac{1}{k^2}$$

and, in general,

$$\frac{A_nB_{n+1}}{B_{n+1}C_n} = \frac{B_nC_{n+1}}{C_{n+1}A_n} = \frac{C_nA_{n+1}}{A_{n+1}B_n} = \begin{cases} k^{2^n} & \text{for } n \text{ even} \\ \frac{1}{k^{2^n}} & \text{for } n \text{ odd.} \end{cases}$$

Prove that $\triangle ABC$ formed by lines A_0A_1, B_0B_1, C_0C_1 is contained in $\triangle A_nB_nC_n$ for any n .

Olympiad 19 (1956)

Tour 19.1

Grade 7

19.1.7.1. Prove that there are no four points A, B, C, D on a plane such that all triangles $\triangle ABC, \triangle BCD, \triangle CDA, \triangle DAB$ are acute ones.

19.1.7.2. Find all two-digit numbers x the sum of whose digits is the same as that of $2x, 3x$, etc., $9x$.

19.1.7.3. A closed self-intersecting broken line intersects each of its segments once. Prove that the number of its segments is even.

19.1.7.4. Find all integers that can divide both the numerator and denominator of the ratio $\frac{5l+6}{8l+7}$ for an integer l .

19.1.7.5. What is the least number of points that can be chosen on a circle of length 1956, so that for each of these points there is exactly one chosen point at distance 1, and exactly one chosen point at distance 2 (distances are measured along the circle)?

Grade 8

19.1.8.1. On sides AB and CB of $\triangle ABC$ there are drawn equal segments, AD and CE , respectively, of arbitrary length (but shorter than $\min(AB, BC)$). Find the locus of midpoints of all possible segments DE .

19.1.8.2. In the decimal expression of a positive number, a , all decimals beginning with the third after the decimal point, are deleted (i.e., we take an approximation of a with accuracy to 0.01 with deficiency). The number obtained is divided by a and the quotient is similarly approximated with the same accuracy by a number b . What numbers b can be thus obtained? Write all their possible values. (Cf. Problem 19.1.9.2, 19.1.10.2.)

19.1.8.3. On a circle of length 15 there are selected n points such that for each of them there is exactly one selected point at distance 1 from it, and exactly one is selected point at distance 2 from it. (All distances are measured along the circle.) Prove that n is divisible by 10. (Cf. Problem 19.1.7.5.)

19.1.8.4. Let a, b, c, d, l be integers. Prove that if the numerator and denominator of the ratio $\frac{al+b}{cl+d}$ are both divisible by k , then so is $ad - bc$. (Cf. Problem 19.1.7.4.)

19.1.8.5. On an infinite sheet of graph paper a table is drawn so that in each square of the table stands a number equal to the arithmetic mean of the four adjacent numbers. Out of the table a piece is cut along the lines of the graph paper. Prove that the largest number on the piece always occurs at an edge; see Fig. 27, where $x = \frac{1}{4}(a + b + c + d)$.

FIGURE 27. (Probl. 19.1.8.5)

FIGURE 28. (Probl. 19.1.10.4)

Grade 9

19.1.9.1. In a convex quadrilateral $ABCD$, consider quadrilateral $KLMN$ formed by the centers of mass of triangles ABC, BCD, DBA, CDA . Prove that the straight lines connecting the midpoints of the opposite sides of quadrilateral $ABCD$ meet at the same point as the straight lines connecting the midpoints of the opposite sides of $KLMN$.

19.1.9.2. In the decimal expression of a positive number, a , all decimals beginning with the third after the decimal point, are deleted (i.e., we take an approximation of a rounding off to 0.001 with deficiency). The number obtained is divided by a and the quotient is similarly approximated with the same accuracy by a number b . What numbers b can be thus obtained? Write all their possible values. (Cf. Problems 19.1.8.2, 19.1.10.2.)

19.1.9.3. See Problem 19.1.8.5.

19.1.9.4. Consider positive numbers h, s_1, s_2 , and a spatial triangle $\triangle ABC$. How many ways are there to select a point D so that the height of tetrahedron $ABCD$ dropped from D equals h , and the areas of faces ACD and BCD equal s_1 and s_2 , respectively?

19.1.9.5. See Problem 19.1.8.4.

Grade 10

19.1.10.1. A square of side a is inscribed in a triangle so that two of the square's vertices lie on the base, and the other two lie on the sides of the triangle. Prove that if r is the radius of the circle inscribed in the triangle, then $r\sqrt{2} < a < 2r$.

19.1.10.2. In the decimal expression of a positive number, a , all decimals beginning with the third after the decimal point, are deleted (i.e., we take an approximation of a with accuracy to 0.0001 with deficiency). The number obtained is divided by a and the quotient is similarly approximated with the same accuracy by a number b . What numbers b can be thus obtained? Write all their possible values. (Cf. Problems 19.1.8.2, 19.1.9.2.)

19.1.10.3. See Problem 19.1.8.4.

19.1.10.4. Given a closed broken line $A_1A_2A_3 \dots A_n$ in space and a plane intersecting all its segments, A_1A_2 at B_1 , A_2A_3 at B_2 , \dots , A_nA_1 at B_n , see Fig. 28, prove that

$$\frac{A_1B_1}{B_1A_2} \cdot \frac{A_2B_2}{B_2A_3} \cdot \frac{A_3B_3}{B_3A_4} \cdot \dots \cdot \frac{A_nB_n}{B_nA_1} = 1. \quad (*)$$

19.1.10.5. Prove that the system of equations

$$\begin{cases} x_1 - x_2 = a, \\ x_3 - x_4 = b, \\ x_1 + x_2 + x_3 + x_4 = 1 \end{cases}$$

has at least one solution in positive numbers if and only if $|a| + |b| < 1$.

Tour 19.2

Grade 7

19.2.7.1. Let O be the center of the circle circumscribed around $\triangle ABC$, let A_1, B_1, C_1 be symmetric to O through respective sides of $\triangle ABC$. Prove that all heights of $\triangle A_1B_1C_1$ pass through O , and all heights of $\triangle ABC$ pass through the center of the circle circumscribed around $\triangle A_1B_1C_1$.

19.2.7.2. Points $A_1, A_2, A_3, A_4, A_5, A_6$ divide a circle of radius 1 into six equal arcs. Ray l_1 from A_1 connects A_1 with A_2 ; ray l_2 from A_2 connects A_2 with A_3 , and so on, ray l_6 from A_6 connects A_6 with A_1 . From a point B_1 on l_1 the perpendicular is dropped to l_6 ; from the foot of this perpendicular another perpendicular is dropped to l_5 , and so on. Let the foot of the 6-th perpendicular coincide with B_1 . Find the length of segment A_1B_1 . (Cf. Problem 19.2.9.5.)

19.2.7.3. 100 numbers (some positive, some negative) are written in a row. All of the following three types of numbers are underlined: 1) every positive number, 2) every number whose sum with the number following it is positive, 3) every number whose sum with the two numbers following it is positive. Can the sum of all underlined numbers be (a) negative? (b) equal to zero?

19.2.7.4. 64 non-negative numbers whose sum equals 1956 are arranged in a square table, eight numbers in each row and each column. The sum of the numbers on the two longest diagonals is equal to 112. The numbers situated symmetrically with respect to any of the longest diagonals are equal. Prove that the sum of numbers in any column is less than 1035. (Cf. Problem 19.2.8.2.)

19.2.7.5*. Assume that the number of a tree's leaves is a multiple of 15. Neglecting the shade of the trunk and branches prove that one can rip off the tree $\frac{7}{15}$ of its leaves so that not less than $\frac{8}{15}$ of its shade remains.

Grade 8

19.2.8.1*. A shipment of 13.5 tons is packed in a number of weightless containers. Each loaded container weighs not more than 350 kg. Prove that 11 trucks each of which is capable of carrying ≤ 1.5 ton can carry this load.

19.2.8.2. 64 non-negative numbers whose sum equals 1956 are arranged in a square table, eight numbers in each row and each column. The sum of the numbers on the two longest diagonals is equal to 112. The numbers situated symmetrically with respect to any of the longest diagonals are equal. Prove that the sum of numbers in any row is less than 518. (Cf. Problem 19.2.7.4.)

19.2.8.3. Find the union of all projections of a given line segment AB to all lines passing through a given point O .

19.2.8.4. See Problem 19.2.7.3.

19.2.8.5*. In a rectangle of area 5 sq. units, 9 rectangles of area 1 are arranged. Prove that the area of the overlap of some two of these rectangles is $\geq \frac{1}{9}$. (Cf. Problem 19.210.2.)

Grade 9

19.2.9.1. See Problem 19.2.8.1.

19.2.9.2. 1956 points are chosen in a cube with edge 13. Is it possible to fit inside the cube a cube with edge 1 that would not contain any of the selected points? (See Fig. 29.)

19.2.9.3. Given three numbers x, y, z denote the absolute values of the differences of each pair by x_1, y_1, z_1 . From x_1, y_1, z_1 form in the same fashion the numbers x_2, y_2, z_2 , etc. It is known that $x_n = x, y_n = y, z_n = z$ for some n . Find y and z if $x = 1$.

FIGURE 29. (Probl. 19.2.9.2)

19.2.9.4. A quadrilateral is circumscribed around a circle. Prove that the straight lines connecting neighboring tangent points either meet on the extension of a diagonal of the quadrilateral or are parallel to it. (See Fig. 30.)

FIGURE 30. (Probl. 19.2.9.4)

19.2.9.5*. Let A, B, C be three nodes of a graph paper. Prove that if $\triangle ABC$ is an acute one, then there is at least one more node either inside $\triangle ABC$ or on one of its sides.

Grade 10

19.2.10.1. n numbers (some positive and some negative) are written in a row. Each positive number and each number whose sum with several of the numbers following it is positive is underlined. Prove that the sum of all underlined numbers is positive. (Cf. Problem 19.2.8.4.)

19.2.10.2. In a rectangle of area 5 sq. units, lie 9 arbitrary polygons each of area 1. Prove that the area of the overlap of some two of these rectangles is $\geq \frac{1}{9}$. (Cf. Problem 19.2.8.5.)

19.2.10.3. See Problem 19.2.9.3.

19.2.10.4*. Prove that if the trihedral angles at each of the vertices of a triangular pyramid are formed by the identical planar angles, then all faces of this pyramid are equal.

19.2.10.5. Find points B_1, B_2, \dots, B_n on the extensions of sides $A_1A_2, A_2A_3, \dots, A_nA_1$ of a regular n -gon $A_1A_2 \dots A_n$ such that $B_1B_2 \perp A_1A_2, B_2B_3 \perp A_2A_3, \dots, B_nB_1 \perp A_nA_1$.

Olympiad 20 (1957)

Tour 20.1

Grade 7

20.1.7.1. Find all isosceles trapezoids that are divided into 2 isosceles triangles by a diagonal.

20.1.7.2. Let $ax^3 + bx^2 + cx + d$ be divisible by 5 for given positive integers a, b, c, d and any integer x . Prove that a, b, c and d are all divisible by 5.

20.1.7.3. A snail crawls over a table at a constant speed. Every 15 minutes it turns by 90° , and in-between these turns it crawls along a straight line. Prove that it can return to the starting point only in an integer number of hours.

20.1.7.4. See Problem 20.1.8.4.

20.1.7.5. The distance between towns A and B is 999 km. At every kilometer of the road that connects A and B a sign shows the distances to A and B as follows:

$$\boxed{0|999} \quad \boxed{1|998} \quad \boxed{2|997} \quad \dots \quad \boxed{998|1} \quad \boxed{999|0}$$

How many signs are there, with both distances written with the help of only two distinct digits?

Grade 8

20.1.8.1. Given two concentric circles and a pair of parallel lines. Find the locus of the fourth vertices of all rectangles with three vertices on the concentric circles, two vertices on one circle and the third on the other and with sides parallel to the given lines. (See Fig. 31.)

FIGURE 31. (Probl. 20.1.8.1)

FIGURE 32. (Probl. 20.1.10.2)

20.1.8.2. See Problem 20.1.7.3.

20.1.8.3. Of all parallelograms of a given area find the one with the shortest possible longer diagonal.

20.1.8.4. For any column and any row in a rectangular numerical table, the product of the sum of the numbers in a column by the sum of the numbers in a row is equal to the number at the intersection of the column and the row. Prove that either the sum of all the numbers in the table is equal to 1 or all the numbers are equal to 0.

20.1.8.5. Let $ax^4 + bx^3 + cx^2 + dx + e$ be divisible by 7 for given positive integers a, b, c, d, e and all integers x . Prove that a, b, c, d and e are all divisible by 7. (Cf. Problem 20.1.7.2.)

Grade 9

20.1.9.1. See Problem 20.1.8.4.

20.1.9.2. Solve the equation $x^3 - [x] = 3$.

20.1.9.3. In a quadrilateral $ABCD$ points M and N are the midpoints of the diagonals AC and BD , respectively. The line through M and N meets AB and CD at M' and N' , respectively. Prove that if $MM' = NN'$, then $AD \parallel BC$.

20.1.9.4. A student takes a subway to an Olympiad, pays one rouble and gets his change. Prove that if he takes a tram (street car) on his way home, he will have enough coins to pay the fare without change.

Note: In 1957, the price of a subway ticket was 50 kopeks, that of a tram ticket 30 kopeks, the denominations of the coins were 1, 2, 3, 5, 10, 15, and 20 kopeks. (1 rouble = 100 kopeks.)

20.1.9.5. See Problem 20.1.10.5.

Grade 10

20.1.10.1. For which integer n is $N = 20^n + 16^n - 3^n - 1$ divisible by 323?

20.1.10.2. The segments of a closed broken line in space are of equal length, and each three consecutive segments are mutually perpendicular. Prove that the number of segments is divisible by 6. (Cf. Problem 20.1.7.3.) See Fig. 32.

20.1.10.3. See Problem 20.1.9.3.

20.1.10.4. A student is going to a club. (S)he takes a tram, pays one ruble and gets the change. Prove that on the way back by a tram (s)he will be able to pay the fare without any need to change. (See Note to Problem 20.1.9.4.)

20.1.10.5. A planar polygon $A_1A_2A_3 \dots A_{n-1}A_n$ ($n > 4$) is made of rigid rods that are connected by hinges. Is it possible to bend the polygon (at hinges only!) into a triangle? (See Fig. 33.)

FIGURE 33. (Probl. 20.1.10.5)

Tour 20.2

Grade 7

20.2.7.1. Straight lines OA and OB are perpendicular. Find the locus of endpoints M of all broken lines OM of length l , which intersect each line parallel to OA or OB at not more than one point.

20.2.7.2. A radio lamp has a 7-contact plug, with the contacts arranged in a circle. The plug is inserted into a socket with 7 holes. Is it possible to number the contacts and the holes so that for any insertion at least one contact would match the hole with the same number? (Cf. Problem Problem 20.2.9.3.)

20.2.7.3. The lengths, a and b , of two sides of a triangle are known. What length should the third side be, in order for the largest angle of the triangle to be of the least possible value?

20.2.7.4. A circle is inscribed in a triangle. The tangent points are the vertices of a second triangle in which another circle is inscribed; its tangency points are the vertices of a third triangle; the angles of this triangle are identical to those of the first triangle. Find these angles.

20.2.7.5. Eight consecutive numbers are chosen from the Fibonacci sequence 1, 2, 3, 5, 8, 13, 21, \dots . Prove that the sequence does not contain the sum of chosen numbers.

Grade 8

20.2.8.1. The lengths, a and b , of two sides of a triangle are known. What length should the third side be in order for the smallest angle of the triangle to be of the greatest possible value? (Cf. Problem 20.2.7.3.)

20.2.8.2. Prove that the number of all digits in the sequence 1, 2, 3, \dots , 10^8 is equal to the number of all zeros in the sequence 1, 2, 3, \dots , 10^9 . (Cf. Problem 20.2.10.4.)

20.2.8.3. Given a point O inside an equilateral triangle $\triangle ABC$. Line OG connects O with the center of mass G of the triangle and intersects the sides of the triangle, or their continuations, at points A' , B' , C' (See Fig. 34.). Prove that

$$\frac{A'O}{A'G} + \frac{B'O}{B'G} + \frac{C'O}{C'G} = 3.$$

FIGURE 34. (Probl. 20.2.8.3)

20.2.8.4. Solve the system:

$$\begin{cases} \frac{2x_1^2}{1+x_1^2} = x_2, \\ \frac{2x_2^2}{1+x_2^2} = x_3, \\ \frac{2x_3^2}{1+x_3^2} = x_1. \end{cases}$$

20.2.8.5. A circle is inscribed in a scalene triangle. The tangent points are vertices of another triangle, in which a circle is inscribed whose tangent points are vertices of a third triangle, in which a third circle is inscribed, etc. Prove that the resulting sequence does not contain a pair of similar triangles. (Cf. Problem 20.2.7.4.)

Grade 9

20.2.9.1. Two rectangles on a plane intersect at eight points. Consider every other intersection point; they are connected with line segments; these segments form a quadrilateral. Prove that the area of this quadrilateral does not vary under translations of one of the rectangles.

20.2.9.2. Find all real solutions of the system :

$$\begin{cases} 1 - x_1^2 = x_2, \\ 1 - x_2^2 = x_3, \\ \dots\dots\dots \\ 1 - x_{98}^2 = x_{99}, \\ 1 - x_{99}^2 = x_1. \end{cases}$$

(Cf. Problem 20.2.10.2.)

20.2.9.3. A radio lamp has a 20-contact plug, with the contacts arranged in a circle. The plug is inserted into a socket with 20 holes. Let the contacts in the plug and the socket be already numbered. Is it always possible to insert the plug so that none of the contacts matches its socket? (Cf. Problem 20.2.7.2.)

20.2.9.4. Represent 1957 as the sum of 12 positive integer summands a_1, a_2, \dots, a_{12} for which the number $a_1! \cdot a_2! \cdot a_3! \cdot \dots \cdot a_{12}!$ is minimal.

20.2.9.5*. Three equal circles are tangent to each other externally and to the fourth circle internally. Tangent lines are drawn to the circles from an arbitrary point on the fourth circle. Prove that the sum of the lengths of two tangent lines equals the length of the third tangent. (Cf. Problem 20.2.8.2.)

Grade 10

20.2.10.1. Given quadrilateral $ABCD$ and the directions of its sides. Inscribe a rectangle in $ABCD$.

20.2.10.2*. Find all real solutions of the system :

$$\begin{cases} 1 - x_1^2 = x_2, \\ 1 - x_2^2 = x_3, \\ \dots\dots\dots \\ 1 - x_{n-1}^2 = x_n, \\ 1 - x_n^2 = x_1. \end{cases}$$

20.2.10.3. Point G is the center of the sphere inscribed in a regular tetrahedron $ABCD$. Straight line OG connecting G with a point O inside the tetrahedron intersects the faces at points A', B', C', D' . Prove that

$$\frac{A'O}{A'G} + \frac{B'O}{B'G} + \frac{C'O}{C'G} + \frac{D'O}{D'G} = 4.$$

(Cf. Problem 20.2.8.3.)

20.2.10.4. Prove that the number of all digits in the sequence $1, 2, 3, \dots, 10^k$ is equal to the number of all zeros in the sequence $1, 2, 3, \dots, 10^{k+1}$.

20.2.10.5. Given n integers $a_1 = 1, a_2, \dots, a_n$ such that

$$a_i \leq a_{i+1} \leq 2a_i \quad (i = 1, 2, 3, \dots, n-1)$$

and whose sum is even, find whether it is possible to divide them into two groups so that the sum of numbers in one group is equal to the sum of numbers in the other group.

Olympiad 21 (1958)

Tour 21.1

Grade 7

21.1.7.1. In the following system :

$$\begin{cases} *x + *y + *z = 0, \\ *x + *y + *z = 0, \\ *x + *y + *z = 0, \end{cases}$$

two players replace the asterisks with numbers doing so in turns, one number each. Prove that the one who starts can always get a system with a nonzero solution.

21.1.7.2. Consider two diameters AB and CD of a circle. Prove that if M is an arbitrary point on the circle, and P and Q are its projections to these diameters, then the length of PQ does not depend on the location of M . (See Fig. 35.)

FIGURE 35. (Probl. 21.1.7.2)

21.1.7.3. How many four-digit numbers from 0000 to 9999 (we complete a one-, two-, or three-digit number to a four-digit one by writing zeros in front of it) are there such that the sum of their first two digits is equal to the sum of their last two digits?

21.1.7.4. Given two points A and B on a plane. Construct a square with A and B on its sides and with the least possible sum of distances of A to the vertices of the square.

21.1.7.5. In the following triangular table

0	1	2	1957	1958
1	3	5	3915	
				

each number (except for those in the upper row) is equal to the sum of the two nearest numbers in the row above. Prove that the lowest number is divisible by 1958.

Grade 8

21.1.8.1. Consider a point O inside $\triangle ABC$ and three vectors of length 1 on rays OA, OB, OC . Prove that the sum of the lengths of these vectors is < 1 .

21.1.8.2. Prove that if one root of the following system with integer coefficients is not an integer, then $p_1 = p_2, q_1 = q_2$:

$$\begin{cases} x^2 + p_1x + q_1 = 0, \\ x^2 + p_2x + q_2 = 0. \end{cases}$$

21.1.8.3. On a circular clearing of radius R grow three pines of the same diameter. The centers of the pines' trunks are the vertices of an equilateral triangle, each at distance $\frac{R}{2}$ from the center of the clearing. Two men are looking for one another. They go around the clearing along its border, starting from diametrically opposite points. They move at the same speed and in the same direction, and cannot see each other.

Can three men see one another if they go around the clearing starting from the points situated at the vertices of an equilateral triangle inscribed in this clearing?

21.1.8.4. See Problem 21.1.9.3 for $n = 1958$.

21.1.8.5*. The length of the projections of a polygon to the OX -axis, the bisector of the first and third coordinate angles, the OY -axis, and the bisector of the second and fourth coordinate angles are equal to 4, $3\sqrt{2}$, 5 and $4\sqrt{2}$, respectively. Prove that the area S of the polygon is ≤ 17.5 .

Grade 9

21.1.9.1. An infinite broken line $A_0A_1 \dots A_n \dots$ on a plane, with right angles between its segments, begins at point A_0 with coordinates $x = 0$, $y = 1$, and circumvents the origin O clockwise.

The first segment of this broken line is of length 2 and is parallel to the bisector of the fourth coordinate angle. Each of the subsequent segments intersects one of the coordinate axes, and has an integer length which is the least length sufficient to intersect the axis. Denote the lengths of OA_n by r_n ; let the sum of the lengths of the first n segments of the broken line be S_n . Prove that there exists an n for which $\frac{S_n}{r_n} > 1958$.

21.1.9.2. What is the greatest number of axes of symmetry that a figure in space might have, if the figure is formed of three straight lines no two of which are parallel or identical?

21.1.9.3. Solve in positive integers

$$1 - \frac{1}{2 + \frac{1}{3 + \frac{1}{4 + \dots + \frac{1}{(n-1) + \frac{1}{n}}}}} = \frac{1}{x_1 + \frac{1}{x_2 + \frac{1}{x_3 + \dots + \frac{1}{x_{n-1} + \frac{1}{x_n}}}}$$

21.1.9.4. A segment of length 3^n is split into three equal parts. The first and third parts are fixed. Each of the fixed segments is split into three equal parts the first and third of which are fixed again, and so on, until we get segments of length 1. The endpoints of all fixed segments are called *fixed points*. Prove that for any integer k such that $1 \leq k \leq 3^n$ there are two fixed points the distance between which is equal to k .

Grade 10

21.1.10.1*. See Problem 21.1.8.5. Prove that the area S of the polygon is ≥ 10 .

21.1.10.2. Prove that $1155^{1958} + 34^{1958} \neq n^2$ for any integer n .

21.1.10.3. See Problem 21.1.9.2.

21.1.10.4. On a table lies a regular 100-gon whose vertices are numbered consecutively: 1, 2, \dots , 100. These numbers were then rewritten, in increasing order, according to the distance of the corresponding vertex from the front edge of the table. If vertices are at the same distance from the edge, the left number is written first, and then the right one. All possible sets of numbers corresponding to different positions of the 100-gon are written out. Calculate the sum of the numbers in the 13-th position from the left in all these sets.

21.1.10.5* (J. Littelwood's problem.) Of four straight lines on a plane no two are parallel and no three meet at one point. Along each line a pedestrian walks at a constant speed. It is known that the first pedestrian meets the second, third and fourth ones, and the second pedestrian meets the third and fourth ones. Prove that the third pedestrian meets the fourth one.

Tour 21.2

Grade 7

21.2.7.1. Prove that on a plane it is impossible to arrange more than 4 convex polygons so that each two of them have a common side.

21.2.7.2. There are two rows of 1's and -1 's, each containing 1958 numbers. At each step one is allowed to change the sign of any 11 numbers of the first row. Prove that after a finite number of steps one could change the first row into the second one.

21.2.7.3. Each face of a cube is pasted over with two equal right triangles with a common hypotenuse, one of them white and the other black. (See Fig. 36.) Is it possible to arrange these triangles so that the sum of the white angles at each vertex of the cube be equal to the sum of the black angles at the same vertex?

21.2.7.4. Prove that $(n!)^2 > n^n$ for $n > 2$.

21.2.7.5. On a piece of graph paper with squares of side 1, an $m \times n$ rectangle is drawn along the lines of the graph. Is it possible to draw inside the rectangle, along the lines of the graph, a broken line passing each vertex of the graph inside or on the boundary of the rectangle exactly once? If this is possible, what is the length of the broken line?

FIGURE 36. (Probl. 21.2.7.3)

Grade 8

21.2.8.1. A polygon (not necessarily convex) is cut out of paper. Through two points on the boundary of the polygon a straight line is drawn. The polygon is folded along this straight line and the two pieces of paper are glued to form a new polygon. Prove that the perimeter of the new polygon does not exceed that of the initial polygon. (See Fig. 37.)

FIGURE 37. (Probl. 21.2.8.1)

21.2.8.2. Prove that for any nonnegative a_1 and a_2 such that $a_1 + a_2 = 1$ there exist nonnegative b_1 and b_2 such that $b_1 + b_2 = 1$ and $(1.25 - a_1)b_1 + 3(1.25 - a_2)b_2 > 1$.

21.2.8.3. Inside $\angle AOB$, a point C is taken. From C perpendiculars are dropped: CD to OA and CE to OB . From D and E , perpendiculars are also dropped: DN to OB and EM to OA . Prove that $OC \perp MN$.

21.2.8.4. Prove that $1^1 \cdot 2^2 \cdot 3^3 \cdot \dots \cdot n^n < n^{n(n+1)/2}$ for $n > 1$.

21.2.8.5. Let a be the greatest number of nonintersecting discs of diameter 1 whose centers are inside a polygon M , and let b be the least number of discs of radius 1 that entirely cover M . Which is greater, a or b ?

Grade 9

21.2.9.1. See Problem 21.2.10.1 below.

21.2.9.2. From a point O draw n rays on a plane so that the sum of all angles formed by pairs of rays (their total is $\frac{1}{2}n(n-1)$) is the greatest possible.

21.2.9.3. A playboard is shaped like a rhombus with an angle of 60° . Each side of the rhombus is divided into 9 parts. Straight lines parallel to the sides and to the smaller diagonal of the rhombus are drawn through the division points thus splitting the playboard into triangular cells. If a chip stands in a cell, we draw three straight lines through the center of this cell parallel to the sides and to the smaller diagonal of the rhombus. We say that the chip *wins* all the cells that these three lines intersect. What is the least number of chips needed to win all cells on the chessboard?

21.2.9.4. Let a be the least number of discs of radius 1 which completely cover a polygon M , and let b be the greatest number of nonintersecting discs of radius 1 with centers inside M . Which is greater, a or b ?

21.2.9.5. A circuit of several resistors connects clamps A and B . Each resistor has an input and an output clamp. What is the least number of resistors needed and what should the principal circuit design be for the circuit not to be short or open if any 9 resistors between A and B break? (A resistor is *broken* if it executes a short or open circuit.)

Grade 10

21.2.10.1. Solve in positive integers

$$x^{2y} + (x + 1)^{2y} = (x + 2)^{2y}.$$

21.2.10.2. In a polygon, there are points A and B such that the length of any broken line connecting them and passing inside or along the boundary of the polygon is > 1 . Prove that the perimeter of the polygon is > 2 .

21.2.10.3. A school curriculum has $2n$ subjects. All students get only A 's and B 's. We will say for the sake of argument that one student is *better* than another if (s)he is not worse than the other in all subjects and better in some subjects. Suppose that no two students get the same grades and it is impossible to say which of any two students is better. Prove that the number of students in this school does not exceed $\binom{2n}{n}$.

21.2.10.4. The lengths of a parallelogram's sides are equal to a and b . Find the ratio of the volumes of bodies obtained by rotating the parallelogram around side a and around side b .

21.2.10.5. We are given n cards with numbers written on them, one number on each side: 0 and 1 on the 1-st, 1 and 2 on the 2-nd, etc., $n - 1$ and n on the n -th card. One person takes several cards and shows to his/her partner one side of these cards. Indicate all the cases in which the second person can determine the number written on the other side of the last card shown to him.

Olympiad 22 (1959)

Tour 22.1

Grade 7

22.1.7.1. Let a and b be integers. Let us fill in two columns as follows. Write a and b in the first row. In the second row write a number a_1 equal to $a/2$ if a is even and $(a - 1)/2$ if a is odd and $b_1 = 2b$. In the third row write a number a_2 equal to $a_1/2$ if a_1 is even and $(a_1 - 1)/2$ if a_1 is odd and $b_2 = 4b$. Continue until you get a 1 in the left column.

Prove that the sum of the b_i for which a_i is odd is equal to ab .

22.1.7.2. Prove that $2^{2^{1959}} - 1$ is divisible by 3.

22.1.7.3*. Is it possible to arrange in a sequence all three-digit numbers that do not end in zeros so that the last digit of each number is equal to the first digit of the number following it?

22.1.7.4. How should a rook move on a chessboard to pass each square once and with the least number of turning points?

22.1.7.5. Given a square of side 1, find the set of points the sum of whose distances to the sides of this square (or their extensions) equals 4.

Grade 8

22.1.8.1. Consider two barrels of sufficient capacity. Find if it is possible to pour exactly 1 liter from one barrel into the other using two containers that can hold $2 - \sqrt{2}$ and $\sqrt{2}$ liters?

22.1.8.2. On a piece of paper, write figures 0 to 9. Observe that if we turn the paper through 180° the 0's, 1's (written as a vertical line segment, not as in the typed texts) and 8's turn into themselves, the 6's and 9's interchange, and the other figures become meaningless.

How many 9-digit numbers are there which turn into themselves when a piece of paper on which they are written is turned by 180° ?

22.1.8.3. Consider a convex quadrangle $ABCD$. Denote the midpoints of AB and CD by K and M , respectively; denote the intersection point of AM and DK by O and that of BM and CK by P . Prove that the area of quadrangle $MOKP$ is equal to the sum of the areas of $\triangle BPC$ and $\triangle AOD$.

22.1.8.4. See Problem 22.1.7.4.

22.1.8.5. Two circles with centers at O_1 and O_2 do not intersect. Let a_1 and a_2 be the inner tangents and a_3 and a_4 the outer tangents to these circles. Further, let a_5 and a_6 be the tangents to the circle with center at O_1 drawn from O_2 ; let a_7 and a_8 be the tangents to the circle with center at O_2 drawn from O_1 . Denote the intersection point of a_1 with a_2 by O .

Prove that it is possible to draw two circles with centers at O so that the first one is tangent to a_3 and a_4 and the second one is tangent to a_5, a_6, a_7, a_8 , and so that the radius of the second circle is half that of the first one.

Grade 9

22.1.9.1. Consider 1959 positive numbers $a_1, a_2, \dots, a_{1959}$ whose sum is equal to 1. Consider all different combinations (subsets) of 1 000 of these numbers. Two combinations are assumed to be identical if they differ only in the order of their elements. For each combination we formed the product of its elements. Prove that the sum of all these products is < 1 .

22.1.9.2. See Problem 22.1.8.2.

22.1.9.3*. Given a circle and two points. Construct a circle passing through the given points and intercepting a chord of given length on the given circle.

22.1.9.4. Consider a sheet of graph paper with squares of side 1, let p_k be the number of all broken lines of length k beginning at a fixed known node O of the graph (all broken lines are constituted by segments of the graph). Prove that $p_k < 2 \cdot 3^k$ for any k .

22.1.9.5*. Prove that there is no tetrahedron such that each its edge is a leg of an obtuse planar angle.

Grade 10

22.1.10.1. Prove that there are no integers x, y, z such that $x^k + y^k = z^k$ for an integer $k > 0$ provided $z > 0, 0 < x < k, 0 < y < k$.

22.1.10.2. See Problem 22.1.8.3.

22.1.10.3. Can there be a tetrahedron each edge of which is a side of an obtuse planar angle? (Cf. Problem 22.1.9.5.)

22.1.10.4. In a square $N \times N$ table, the numbers 1 to N^2 are written in the following way: 1 can stand at any place, 2 can occupy the row with the same index as that of the column containing 1, number 3 can occupy the row with the same number as that of the column containing 2, etc. What is the difference between the sum of the numbers in the row containing 1 and the sum of the numbers in the column containing N^2 ?

22.1.10.5. Consider a sequence $a_1 \geq a_2 \geq a_3 \geq \dots \geq a_n \geq \dots$ of positive numbers such that

$$a_1 = \frac{1}{2k}; \quad a_1 + a_2 + \dots + a_n + \dots = 1.$$

Prove that there are k numbers in the sequence such that the least of these k number is greater than half the greatest.

*Tour 22.2**Grade 7*

22.2.7.1. For $a_1 > a_2 > \dots > a_n$ and $b_1 > b_2 > \dots > b_n$ prove that

$$a_1 b_1 + a_2 b_2 + \dots + a_n b_n > a_1 b_n + a_2 b_{n-1} + \dots + a_n b_1.$$

22.2.7.2. Given $\triangle ABC$, find a point whose reflection through any side of the triangle lies on the circumscribed circle.

22.2.7.3. What should 999 999 999 be multiplied by to get a number whose decimal expression contains only 1's?

22.2.7.4. Prove that the digits of any six-digit number can be permuted so that the difference between the sum of the first and the last three digits of the new number is less than 10.

22.2.7.5. Consider n numbers x_1, \dots, x_n each equal to 1 or -1 . Prove that if

$$x_1 x_2 + x_2 x_3 + \dots + x_{n-1} x_n + x_n x_1 = 0,$$

then n is divisible by 4.

Grade 8

22.2.8.1. See Problem 22.2.7.5. This problem can be reformulated in a "romantic" way: some of n knights sitting at a round table are enemies. The number of knights whose left neighbors are their friends is equal to the number of knights whose left neighbors are their enemies. Prove that $n \div 4$.

22.2.10.4. Given n complex numbers c_1, \dots, c_n that being represented as points on a complex line are the vertices of a convex n -gon. Prove that if

$$\frac{1}{z - c_1} + \frac{1}{z - c_2} + \dots + \frac{1}{z - c_n} = 0,$$

then the point z is inside this n -gon.

22.2.10.5. Two discs of different diameters are divided into $2n$ equal sectors each, and each sector is painted white or black so that each disc has n white sectors and n black sectors. If the two discs are fixed by a pin piercing their centers, it turns out that the circle bounding the smaller disc is painted twice: on the inside (as part of the small disc) and on the outside (as part of the large disc). Thus, some parts of the circle are painted different colors, and the other parts are of the same color on both sides.

Prove that it is possible to rotate the smaller disc so that the parts painted differently will constitute no less than half of the circle's length.

Olympiad 23 (1960)

Tour 23.1

Grade 7

23.1.7.1. Indicate all amounts of roubles that may be changed with the help of both an even and an odd number of bills.

Remark. We assume that, as it was in reality in 1960, the bills are of denominations of 1, 3, 5, 10, 25, 50 and 100 roubles.

23.1.7.2. Three equal circles with centers O_1, O_2, O_3 intersect at a given point, let A_1, A_2, A_3 be the other intersection points. Prove that $\triangle O_1O_2O_3 = \triangle A_1A_2A_3$.

23.1.7.3. 30 undergraduates from 1-st through 5-th year took part in compiling 40 problems for an Olympiad. Any 2 students of the same year brought about the same number of problems. Any two undergraduates of different years suggested distinct number of problems. How many undergraduates suggested one problem each?

23.1.7.4. Two circles with centers O_1 and O_2 intersect at points M and N . Line O_1M intersects the first circle at A_1 , and the second one at A_2 . Line O_2M intersects the first circle at B_1 , and the second one at B_2 . Prove that A_1B_1, A_2B_2 , and MN intersect at one point. (See Fig. 39.)

23.1.7.5. Prove that an integer n cannot have more than $2\sqrt{n}$ divisors.

Grade 8

23.1.8.1. Prove that a number whose decimal expression contains 300 digits 1, all other digits being zeros, is not a perfect square.

23.1.8.2. In a tournament, each chessplayer got half of his (her) final score in matches with participants who occupied three last places. How many persons participated in the tournament?

23.1.8.3. Draw a straight line through a given vertex A of a convex quadrilateral $ABCD$ so that it divides $ABCD$ into parts of equal area.

23.1.8.4. There are given segments AB, CD and a point O such that no three of the points A, B, C, D, O are on one straight line. The endpoint of a segment is *marked* if the straight line passing through it and O does not intersect another segment. How many marked endpoints are there?

23.1.8.5*. Prove that there are infinitely many positive integers not representable as $p + n^{2k}$ for any prime p and positive integers n and k .

Grade 9

23.1.9.1. Prove that any proper fraction can be represented as a (finite) sum of the reciprocals of distinct integers.

23.1.9.2. See Problem 23.1.8.5.

23.1.9.3. Given a convex polygon and a point O inside it such that any straight line through O divides the polygon's area in halves. Prove that the polygon is symmetric with respect to O .

23.1.9.4. Given a circle and a point inside it. Find the locus of fourth vertices of rectangles, two of whose vertices lie on the given circle and a third vertex is the given point.

Grade 10

23.1.10.1. Two equal equilateral triangular laminas are arranged in space on parallel planes P_1 and P_2 so that the segment connecting their centers is perpendicular to their planes. Find the locus of midpoints of all segments connecting points of one lamina with the points of the other.

23.1.10.2. Prove that if the fraction $\frac{a^n + b^n}{a + b}$ is an integer for positive integers a, b, n , and both the numerator and the denominator of this fraction are divisible by n , then so is the fraction itself.

23.1.10.3. See Problem 23.1.9.4.

23.1.10.4. In the decimal expression of an integer A all digits except the first and the last are zeros; the first and the last are not zeros; and the number of digits is not less than three. Prove that A is not a perfect square.

23.1.10.5. Given numbers a_1, a_2, \dots, a_k such that $a_1^n + a_2^n + \dots + a_k^n = 0$ for any odd n , prove that nonzero of the numbers a_1, \dots, a_k can be combined in pairs consisting of two opposite numbers, i.e., a and $-a$.

Tour 23.2

Grade 7

23.2.7.1. Given four points, A, B, C, D on a plane. Find a point O such that the sum of the distances from O to the given points is the least possible.

23.2.7.2. Prove that a trapezoid can be constructed from the sides of any quadrilateral.

23.2.7.3. Prove that any nonselfintersecting pentagon is situated on one side of at least one of its edges.

23.2.7.4. One year a Sunday never fell on a certain date in any month. Find this date. (A *date* here is a number $n, 1 \leq n \leq 31$).

Grade 8

23.2.8.1. For what smallest n can n points be arranged on a plane so that every 3 of them are the vertices of a right triangle?

23.2.8.2. On an infinite chessboard, denote by (a, b) the square at the intersection of the a -th row and the b -th column. A piece may move from square (a, b) to any of the 8 squares $(a \pm m, b \pm n)$ or $(a \pm n, b \pm m)$, where m and n are fixed numbers. We know that the piece returns to its starting point after x moves. Prove that x is even.

23.2.8.3. See Problem 23.2.7.2.

23.2.8.4*. A snail crawls along a straight line, always forward, at a variable speed. Several observers in succession follow its movements during 6 minutes. Each person begins to observe before the preceding observer finishes the observation and observes the snail for exactly one minute. Each observer noticed that during his (her) minute of observation the snail has crawled exactly 1 meter. Prove that during 6 minutes the snail could have crawled at most 10 meters.

23.2.8.5. Given pentagon $ABCDE$ in which $AB = BC = CD = DE$ and $\angle B = \angle D = 90^\circ$. Prove that a plane may be tiled with such pentagons without gaps or overlaps.

Grade 9

23.2.9.1. We are given m points; each of them are connected with line segments to l points. What values can l take?

23.2.9.2. We are given an arbitrary centrally-symmetric hexagon on whose sides equilateral triangles are constructed outward. Prove that the midpoints of the segments connecting the vertices of neighboring triangles are vertices of a regular hexagon.

23.2.9.3. Prove that on any rectangular chessboard 4 squares wide a knight cannot pass each square exactly once and return in the last move to its starting position.

23.2.9.4. Find the locus of the centers of all rectangles circumscribed around a given acute triangle.

23.2.9.5*. In a square of side 100, N circles of radius 1 are arranged so that any segment of length 10 lying inside the square intersects at least one circle. Prove that $N \geq 400$.

Grade 10

23.2.10.1. The number A is divisible by 2, 3, \dots , 9. Prove that if $2A$ is represented as the sum $2A = a_1 + a_2 + \dots + a_k$ of positive integers each less than 10, then it is possible to select from a_1, a_2, \dots, a_k certain numbers so that the sum of numbers selected is equal to A .

23.2.10.2. A $6n$ -digit number is divisible by 7. The last digit is moved to the beginning of the decimal expression. Prove that the number thus obtained is also divisible by 7.

23.2.10.3. At a gathering of n people, every two persons have two common acquaintances, and every two acquaintances have no common acquaintances. Prove that each of persons present has the same number of acquaintances.

23.2.10.4. See Problem 23.2.9.4.

23.2.10.5. A snail has to crawl $2n$ units along the lines of a piece of graph paper, starting and finishing at a given crossing. Prove that the number of possible routes the snail can take is equal to $\binom{2n}{n}^2$.

Olympiad 24 (1961)

Tour 24.1

Grade 7

24.1.7.1. See Problem 24.1.9.3 below for an even n .

24.1.7.2. Given a 3-digit number \overline{abc} . We take the number \overline{cba} , and subtract the smaller from the greater to get the number $\overline{a_1b_1c_1}$; we perform the same operation with it, and so on (the case $a_1 = 0$ is allowed). Prove that at some step we get either 495 or 0.

24.1.7.3. Given an acute triangle $\triangle A_0B_0C_0$ let points A_1, B_1, C_1 be the centers of squares constructed on sides B_0C_0, C_0A_0, A_0B_0 outwards. We take triangle $\triangle A_1B_1C_1$, perform the same operation with it and get $\triangle A_2B_2C_2$, etc. Prove that $\triangle A_nB_nC_n$ and $\triangle A_{n+1}B_{n+1}C_{n+1}$ intersect in exactly 6 points.

24.1.7.4. Consider 100 points on a plane such that (1) the distance between any two of them does not exceed 1 and (2) if A, B, C are any three of these points, then $\triangle ABC$ is obtuse. Prove that there is a circle of radius $1/2$ such that all given points are either inside it or on it.

24.1.7.5*. On a chessboard, two squares of the same color are selected. Prove that a rook can traverse all squares, starting from one of those selected, and visiting each square exactly once except for the other selected square which the rook must visit twice.

Grade 8

24.1.8.1. Consider $\triangle ABC$ and a point O , denote by M_1, M_2, M_3 the centers of mass of $\triangle OAB, \triangle OBC, \triangle OCA$, respectively. Prove that $S_{M_1M_2M_3} = \frac{1}{9}S_{ABC}$.

24.1.8.2. One of two players selects a set of one-digit numbers x_1, \dots, x_n (either all positive or all negative). The second player can ask what is the value of $a_1x_1 + a_2x_2 + \dots + a_nx_n$, where a_1, \dots, a_n are any numbers the second player wishes. What is the least number of questions the second player can use to guess what is the selected set of x 's?

24.1.8.3. See Problem 24.1.7.3.

24.1.8.4. Prove that a rook can pass all squares of a rectangular chessboard visiting each square exactly once and return to the first square only if the number of squares is even.

24.1.8.5. A set of consecutive positive integers $a, a + 1, \dots, a + k$ is called a *segment* (of the natural series). Two segments of length 1961 are written one below the other. Prove that it is possible to arrange the numbers of each segment so that by adding digits which stand one below the other we get another segment.

Grade 9

24.1.9.1. See Problem 24.1.7.1.

24.1.9.2. See Problem 24.1.8.2.

24.1.9.3. Prove that it is possible to arrange the numbers from 1 to n^2 in an $n \times n$ table so that the sums of numbers in each column are equal.

24.1.9.4. See Problem 24.1.10.3 below assuming that k is divisible by 4.

24.1.9.5. On a plane there are n points such that if A, B, C are any three of them, no other point is inside $\triangle ABC$. Prove that these points may be numbered so that the polygon $A_1A_2 \dots A_n$ is convex.

Grade 10

24.1.10.1. Given the Fibonacci sequence $1, 1, 2, 3, 5, \dots, u_k, \dots$. Prove that u_{5k} is divisible by 5 for any $k = 1, 2, 3, \dots$

24.1.10.2. On a plane several strips of different width are drawn so that no two of them are parallel. How should the strips be transported parallel with themselves to maximize the area of their intersection F ? (See Fig. 40.)

FIGURE 40. (Probl. 24.1.10.2)

FIGURE 41. (Probl. 24.2.7.1)

24.1.10.3. k persons took a bus without a conductor. They had only coins of denomination 10, 15, or 20 kopecks. Each person paid his/her fare and got the change from other passengers. Prove that the least number of coins needed for this operation is equal to $k + \left\lceil \frac{k+3}{4} \right\rceil$.

Remark. Recall that the machines that sold tickets in the public transport in Moscow were self service. They had receptacles (cash-boxes) for any amount of money but could not give any change. The bus fare was 5 kopecks. So if you had just a 10 kopeck coin you had to ask someone in need of a ticket to give you his/her 5 kopecks, insert your 10 kopecks and take 2 tickets. So the passengers had to help each other or risk a fine¹.

24.1.10.4. A circle S and a point O outside it are both on the same plane. Consider an arbitrary sphere through S and the cone with vertex at O tangent to this sphere. Find the locus of the centers of all circles along which such cones are tangent to such spheres.

24.1.10.5. Given n nonzero complex numbers z_i , $i = 1, \dots, n$, such that $z_1 + z_2 + \dots + z_n = 0$, prove that among them there are two numbers with the difference between their arguments $\geq 120^\circ$.

Tour 24.2

Grade 7

24.2.7.1. The sides of an arbitrary convex polygon are painted on the outside. Consider several diagonals; let each of them be similarly painted on one side. Prove that at least one of the polygons into which the initial one is divided by the diagonals is painted completely on the outside. (We allow the paint to leak inside a polygon at its vertices.) See Fig. 41.

24.2.7.2. On sides AB , BC , CD and AD of square $ABCD$ points P , Q , R , S , respectively, are selected so that $PQRS$ is a rectangle. Prove that either $PQRS$ is a square or its sides are parallel to the respective diagonals of $ABCD$.

24.2.7.3. Prove that among any 39 consecutive positive integers there is at least one the sum of whose digits is divisible by 11.

24.2.7.4. Given a 4×4 table. Show that it is possible to arrange 7 asterisks in the table's squares so that if we strike out any two rows and any two columns the remaining squares still contain at least one asterisk. Prove that if there are fewer than 7 asterisks it is always possible to strike out two rows and two columns with no asterisks remaining.

24.2.7.5. Prove that the following system has no integer solutions for a, b, c, d

$$\begin{cases} abcd - a = 1961, \\ abcd - b = 961, \\ abcd - c = 61, \\ abcd - d = 1. \end{cases}$$

¹One could actually take as many tickets as there were in the machine; it was only your conscious and, perhaps, the presence of other passengers, if any, that prevented you from abuse. Miraculously, this seldom happened.

Grade 8

24.2.8.1. Given a figure of 16 segments, see Fig. 42. Prove that it is impossible to draw a broken line intersecting each of the segments exactly once. The broken line may be open and selfintersecting but its vertices may not lie on the segments and its links may not pass through the common endpoints of the given segments.

FIGURE 42. (Probl. 24.2.8.1)

24.2.8.2*. The length of a diagonal of a rectangle is equal to d . The rectangle's vertices are the centers of 4 circles of radii r_1, r_2, r_3, r_4 such that $r_1 + r_3 = r_2 + r_4 < d$. Two pairs of outer tangents to circles 1, 3 and 2, 4, are drawn. Prove that a circle can be inscribed in the quadrilateral formed by these four tangents. (See Fig. 43.)

FIGURE 43. (Probl. 24.2.8.2)

24.2.8.3. The sum of digits of integers k and $k + l$ is divisible by 11 and there is no number with similar properties between them. What is the greatest value of l ? (Cf. Problem 24.2.7.3.)

24.2.8.4. See Problem 24.2.7.4.

24.2.8.5. Given four numbers, a, b, c, d , we construct another four numbers: ab, bc, cd, da (each number is multiplied by the next one and the fourth number is multiplied by the first one). From these four numbers a third foursome is obtained by the same rule, etc. Prove that in the resulting sequence of foursomes we never encounter the initial one except for the case $a = b = c = d = 1$.

Grade 9

24.2.9.1. Points A and B move uniformly with equal angle velocities clockwise along circles O_1 and O_2 , respectively. Prove that vertex C of equilateral triangle $\triangle ABC$ also moves uniformly along a circle.

24.2.9.2. An $m \times n$ table is filled with certain numbers. It is allowed to simultaneously change the sign of all numbers in a column or a row. Prove that by applying this operation several times, any given table may be altered so that the sum of the numbers in any one of its columns or rows will be nonnegative.

24.2.9.3. n points are connected by segments so that each point is connected to any other by a "route", and no two points are connected by more than one such "route". Prove that there are $n - 1$ segments altogether.

24.2.9.4. a, b, p are integers. Prove that there exist relatively prime integers k, l such that $ak + bl$ is divisible by p .

24.2.9.5. Nick and Pete divide between themselves $2n + 1$ nuts, $n \geq 2$, and each tries to get the greater share, naturally. According to The Rule there are three ways to divide the nuts. Each way consists of three steps and the 1-st and 2-nd steps are common for all three ways.

1-st step: Pete divides all nuts into two piles, each containing no less than two nuts.

2-nd step: Nick divides both piles into two, each new pile containing no less than one nut.

3-rd step: Nick sticks to either of the following methods:

- a) Nick takes either the biggest and the smallest pile, or
- b) Nick takes both medium-sized piles, or
- c) Nick chooses either the biggest and the smallest or the medium-sized piles, but pays Pete one nut for the choice.

Find the most profitable and the least profitable of methods a) – c) for Nick to divide the nuts.

Grade 10

24.2.10.1. Prove that for any three infinite sequences of positive integers

$$a_1, a_2, \dots, a_n, \dots;$$

$$b_1, b_2, \dots, b_n, \dots;$$

$$c_1, c_2, \dots, c_n, \dots$$

there exist p and q such that $a_p \geq a_q$, $b_p \geq b_q$, $c_p \geq c_q$.

24.2.10.2. 120 squares of side 1 are tossed onto a 20×25 rectangle. Prove that a disc of diameter 1 can be placed in the rectangle so that the disc does not intersect any of the squares.

24.2.10.3. See Problem 24.2.9.2.

24.2.10.4. On a plane, the distance from a fixed point P to two vertices, A and B , of an equilateral $\triangle ABC$ is 2 and 3 units, respectively. Find the maximal possible length of PC .

24.2.10.5. From an arbitrary sequence of 2^k numbers 1 and -1 we get a new sequence by the following operation: each number is multiplied by the one following it, and the last 2^k -th number is multiplied by the 1-st one. We perform the same operation with the sequence obtained, and so on. Prove that eventually we get a sequence consisting entirely of 1's.

Olympiad 25 (1962)

Tour 25.1

Grade 7

25.1.7.1. Given a straight line l perpendicular to and intersecting segment AB . For any point M on l we can find a point N such that $\angle NAB = 2\angle MAB$ and $\angle NBA = 2\angle MBA$. Prove that the absolute value $|AN - BN|$ does not depend on M . (See Fig. 44.)

FIGURE 44. (Probl. 25.1.7.1)

25.1.7.2. We reflect an equilateral triangle with one marked side through one of its sides. Then we similarly reflect the resulting triangle, etc., until at a certain step the triangle returns to its initial position. Prove that the marked side also returns to its initial position.

25.1.7.3. Let a, b, c, d be the sides of a quadrilateral that is not a rhombus. Prove that from the segments a, b, c, d one can construct a self-intersecting quadrilateral.

25.1.7.4. Denote by $S(a)$ the sum of digits of a number a . Prove that if $S(a) = S(2a)$, then a is divisible by 9.

25.1.7.5. On each side of n given cards one of the numbers $1, 2, \dots, n$ is written so that each number occurs exactly twice. Prove that the cards may be arranged on a table so that all numbers $1, 2, \dots, n$ face upward.

Grade 8

25.1.8.1. On sides AB, BC, CA of an equilateral triangle $\triangle ABC$ find points X, Y, Z , respectively, so that the area of the triangle formed by lines CX, BZ, AY is one-fourth of the area of $\triangle ABC$ and so that

$$\frac{AX}{XB} = \frac{BY}{YC} = \frac{CZ}{ZA}.$$

25.1.8.2. See Problem 25.1.7.2.

25.1.8.3. Prove that for any integer d there exist integers m and n such that

$$d = \frac{n - 2m + 1}{m^2 - n}.$$

25.1.8.4. See Problem 25.1.7.4.

25.1.8.5. See Problem 25.1.7.5.

Grade 9

25.1.9.1. Given two intersecting segments AA_1 and BB_1 on which lie points M and N , respectively, so that $AM = BN$. Find positions of M and N for which the length of MN is the shortest. (Cf. Problem 25.1.9.2.7-8.3).

25.1.9.2. A chessman that crosses n squares in one move diagonally and 1 square up (or the other way round) is called a *Boo*. A Boo stands on a square of an infinite chessboard. What n is required for the Boo to reach any given square? For what n this is impossible?

25.1.9.3. See Problem 25.1.7.4.

25.1.9.4. Given the system of equations:

$$\begin{cases} x_1 x_2 x_3 \dots x_{1961} x_{1962} = 1, \\ x_1 - x_2 x_3 \dots x_{1961} x_{1962} = 1, \\ x_1 x_2 - x_3 \dots x_{1961} x_{1962} = 1, \\ x_1 x_2 x_3 \dots x_{1961} - x_{1962} = 1, \end{cases}$$

find what values x_{25} can take.

25.1.9.5. Prove that in a rectangle of area 1 nonintersecting circles can be arranged so that the sum of their radii is equal to 1962.

Grade 10

25.1.10.1. See Problem 25.1.9.1, the segments being replaced with intersecting rays. (See Fig. 45.)

FIGURE 45. (Probl. 25.1.10.1)

FIGURE 46. (Probl. 25.1.10.2)

25.1.10.2. The sides of a square are the bases of equal acute isosceles triangles constructed outward. Prove that the figure obtained cannot be divided into parallelograms. (See Fig. 46.)

25.1.10.3. Prove that any positive integer can be represented as the sum of several distinct terms of the Fibonacci sequence 1, 2, 3, 5, 8, 13,

25.1.10.4. See Problem 25.1.9.4.

25.1.10.5. See Problem 25.1.7.5.

Tour 25.2

Grade 7

25.2.7.1. A ball rests at the side of a billiard table shaped in the form of a regular $2n$ -gon without pockets. How should the ball be shot so that after reflections through all sides¹ (*except the initial one*) *exactly once* it returns to the same point? Prove that *for the paths that consecutively reflect themselves through neighboring sides* the length of the ball's path does not depend on the starting point.

25.2.7.2*. Let $\triangle ABC$ be an isosceles triangle, $AB = BC$, BH its height, M the midpoint of AB , and K the other intersection point of BH with the circle drawn through B , M and C . Prove that $BK = 3R/2$, where R is the radius of the circle circumscribed around $\triangle ABC$.

25.2.7.3. An L-shaped figure (see Fig. 47) is constructed of three squares with side 1. Prove that a) it is impossible to split a rectangle of size 1961×1963 into such figures but b) it is possible to do so with a rectangle of size 1963×1965 .

FIGURE 47. (Probl. 25.2.7.3)

FIGURE 48. (Probl. 25.2.8.5)

25.2.7.4. Prove that the number 100...01 with 1961 zeros between the 1's is not a prime.

25.2.7.5. Given 25 points on a plane such that from any three points we can choose two points that are less than 1 unit of length apart. Prove that 13 of the given points lie on a unit disc.

Grade 8

25.2.8.1. Several diagonals in a convex polygon satisfy the following condition: no two of them intersect except at an endpoint identical with a vertex. Prove that no diagonal come out of at least 2 vertices of this polygon.

25.2.8.2. How should one arrange the numbers 1, 2, ..., 1962 in a sequence $a_1, a_2, \dots, a_{1962}$ in order to obtain the greatest possible value of the sum

$$|a_1 - a_2| + |a_2 - a_3| + \dots + |a_{1961} - a_{1962}| + |a_{1962} - a_1| ?$$

25.2.8.3. An irregular n -gon is inscribed in a circle. After a rotation of the circle around its center through an angle of $\alpha \neq 2\pi$ the n -gon coincides with itself. Prove that n is not prime.

25.2.8.4*. From the numbers x_1, x_2, x_3, x_4, x_5 ten sums are composed each having as summands two of these numbers. Denote the sums by a_1, a_2, \dots, a_{10} ; we do not know what summands constitute them. Prove that given a_1, a_2, \dots, a_{10} , one can find x_1, x_2, \dots, x_5 .

25.2.8.5. Two circles, O_1 and O_2 , intersect at M and P . Denote by MA the chord of O_1 tangent to O_2 at M , and by MB the chord of O_2 tangent to O_1 at M . On line MP , segment PH equal to MP is constructed. Prove that quadrilateral $MAHB$ can be inscribed in a circle. (See Fig. 48.)

¹The words in italics are added to the original formulation to make the problem correct.

Grade 9

25.2.9.1. During every period of 7 consecutive days throughout the school year a student must solve exactly 25 problems. The time required to solve any (one) problem does not vary during a day but does vary during the year according to a Rule known to the student. This time is always less than 45 minutes. The student wants to spend as little time as possible on solving all problems. Prove that to this end (s)he can choose a certain day every week and solve all 25 problems during this day.

Remark. We disregard here the fact that unless the student is looking for trouble at school and at home (s)he can be engaged in solving the problems only on Sundays, for about 18 hours in a row.

25.2.9.2. See Problem 25.2.8.2, where 25 arbitrary different numbers replace 1, 2, ..., 1962.

25.2.9.3. The sides of a convex polygon whose perimeter is equal to 12 are moved a distance of $d = 1$ outward and their extensions form a new polygon. Prove that the area of the new polygon is at least 15 square units greater than the area of the original polygon.

25.2.9.4. See Problem 25.2.8.4.

25.2.9.5. Given 2^n finite sequences of 0's and 1's such that none of them is the beginning of another, prove that the sum of the lengths of these sequences is not less than $n2^n$.

Grade 10

25.2.10.1. A point C is fixed on a given straight line l passing through the center O of a given circle. Points A and A' lie on the circle on one side of l , so that the angles formed by lines AC and $A'C$ with l are equal. Lines AA' and l meet at B . Prove that the location of B does not depend on that of A . (See Fig. 49.)

FIGURE 49. (Probl. 25.2.10.1)

FIGURE 50. (Probl. 25.2.10.4)

25.2.10.2. See Problem 25.2.9.2.

25.2.10.3. See Problem 25.2.9.3.

25.2.10.4. How should a right parallelepiped be placed in space so that the area of its projections to the horizontal plane is the greatest possible? (See Fig. 50.)

25.2.10.5. In a chess tournament, each participant played one game with each other. Prove that the participants may be so numbered, that none of them loses to the one with the next number.

Olympiad 26 (1963)*Tour 26.1**Grade 7*

26.1.7.1. From vertex B of an arbitrary $\triangle ABC$, straight lines BM and BN are drawn outside the triangle so that $\angle ABM = \angle CBN$. Points A' and C' are symmetric to A and C through BM and BN , respectively. Prove that $AC' = A'C$. (See Fig. 51.)

26.1.7.2. Let a, b, c be three numbers such that $a + b + c = 0$. Prove that $ab + bc + ca \leq 0$.

FIGURE 51. (Probl. 26.1.7.1)

26.1.7.3. We have a 4×100 sheet of graph paper. The Rule allows to divide it into 200 rectangular cards of size 1×2 each consisting of 2 cells of the paper and write 1 on one cell of the card and -1 on the other. Is it possible to ensure that the products of the numbers in each column and each row of the table obtained are positive? (Cf. Problem 26.1.8.5 below.)

26.1.7.4. See Problem 26.1.8.4 below.

26.1.7.5. Is it possible to draw a straight line on a 20×30 piece of graph paper so that it would intersect 50 squares? (Cf. Problem 26.1.10.3 below.)

Grade 8

26.1.8.1. Let a_1, \dots, a_n be numbers such that $a_1 + a_2 + \dots + a_n = 0$. Let S be the sum of all products $a_i a_j$ for $i \neq j$. Prove that $S \leq 0$. (Cf. Problem 26.1.7.2.)

26.1.8.2. Given a convex quadrilateral $ABCD$ of area S , a point M inside it and points E, F, G, H symmetric to M through the midpoints of the sides of the quadrilateral $ABCD$, respectively, find the area of quadrilateral $EFGH$. (See Fig. 52.)

FIGURE 52. (Probl. 26.1.8.2)

FIGURE 53. (Probl. 26.1.9.5)

26.1.8.3. Solve in integers the equation

$$\frac{xy}{z} + \frac{xz}{y} + \frac{yz}{x} = 3.$$

26.1.8.4. Given 7 lines on a plane, no two of which are parallel, prove that two of them meet at an angle $< 26^\circ$.

26.1.8.5. A $5 \times n$ piece of graph paper is divided into rectangular 1×2 cards of two cells of the paper each. We write a 1 on one cell of the card and a -1 on the other cell. It is known that the product of the numbers in each row and each column of the resulting table is positive. For which n this is possible? (Cf. Problem 26.1.7.3.)

Grade 9

26.1.9.1. The first term and the difference of an arithmetic progression are integers. Prove that there exists a term in this progression whose decimal expression contains figure 9.

26.1.9.2. See Problem 26.1.8.5.

26.1.9.3. Let a, b, c be some positive numbers. Prove that

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2}.$$

26.1.9.4. Prove that of any four points on a plane, no three of which are on the same line, three points may be selected so that the triangle with vertices at these points has at least one angle $\leq 45^\circ$. (Cf. Problem 26.1.10.2 below.)

26.1.9.5. Is it possible to inscribe in a rectangle with the ratio of sides 9 : 16 another rectangle, with the ratio of sides 4 : 7, so that on each side of the first rectangle there is a vertex of the second one? (See Fig. 53.)

Grade 10

26.1.10.1. See Problem 26.1.9.1.

26.1.10.2. Prove that of any six points in a plane, no three of which are on the same line, three points may be chosen so that the triangle with vertices at these points has at least one angle that is not greater than 30° .

26.1.10.3. What is the greatest number of squares that a line drawn on an $m \times n$ piece of graph paper can intersect?

26.1.10.4. Given numbers a, b, c such that $abc > 0$ and $a + b + c > 0$, prove that $a^n + b^n + c^n > 0$ for any positive integer n .

26.1.10.5. Given an arbitrary $\triangle ABC$, find the locus of points M such that the perpendiculars to lines AM, BM, CM dropped from points A, B, C , respectively, meet at one point.

Grade 11

26.1.11.1. Prove that $x + y + z > xyz$ if $x, y, z > 0$ and

$$\arctan x + \arctan y + \arctan z < \pi.$$

26.1.11.2. Consider a system of 25 distinct segments with a common endpoint at point A and other endpoints lying on a line l not passing through A . Prove that there does not exist a closed 25-angled broken line each of whose segments is parallel and equal to one of the segments from the system considered.

26.1.11.3. See Problem 26.1.10.5.

26.1.11.4. Prove that the sum of all possible 7-digit numbers in whose decimal expression each of the figures 1, 2, 3, 4, 5, 6, 7 is used exactly once is divisible by 9.

26.1.11.5. Each edge of a regular tetrahedron is divided into three equal parts. Through each division point two planes are drawn parallel to the two faces of the tetrahedron that do not pass through this point. Into how many parts do these planes divide the tetrahedron?

Tour 26.2

Grade 7

26.2.7.1. A factory produces rattles shaped in the form of a ring with 3 red and 7 blue spherical beads on it. Two rattles are said to be *of the same type* if one can be obtained from the other one by moving a bead along the ring or by flipping the ring over in space. How many different types of rattles can be manufactured?

26.2.7.2. See Problem 26.2.9.2.

26.2.7.3. Given $\triangle ABC$. Consider straight line intersecting sides AB and AC of the triangle so that the distance from the line to point A is equal to the sum of the distances from the line to points B and C . Prove that all such lines pass through one point.

26.2.7.4. What greatest number of elements can be selected from the set of numbers 1, 2, ..., 1963 so that the sum of any two of the selected numbers is divisible by 26?

26.2.7.5. A system of segments is called *connected* if from the endpoints of any segment any of endpoints of any other segment can be reached by moving along the segments. We assume that it is impossible to pass from one segment to another one at intersection points other than those of connection. Is it possible to connect five points by segments into a connected system so that after erasing any of its segments one gets exactly two connected systems of segments, disconnected from each other?

Grade 8

26.2.8.1. Let a_1, \dots, a_n be arbitrary positive integers. Denote by b_k the number of integers that satisfy $a_i \geq k$. Prove that

$$a_1 + a_2 + \dots + a_n = b_1 + b_2 + \dots .$$

26.2.8.2. An 8×8 table contains all integers from 1 to 64. The numbers are called *adjacent* if the squares they are written upon have a common side. Prove that there exist two adjacent numbers whose difference is not less than 5.

26.2.8.3. Find the set of the centers of mass of all acute triangles inscribed in a given circle.

26.2.8.4. What greatest number of integers can be selected from the set $1, 2, \dots, 1963$ so that no sum of any two selected numbers were divisible by their difference?

26.2.8.5*. Three gentlemen walk along a path 100 meters long at a constant speed of 1, 2, and 3 km/hr, respectively. Reaching the end of the path each of them turns and goes back at the same speed. Prove that there is an interval of 1 minute during which all three gentlemen walk in the same direction.

Grade 9

26.2.9.1*. Given an arbitrary $\triangle ABC$, its medians AM, BN, CQ , and a point X outside it. Prove that the area of one of the triangles $\triangle XAM, \triangle XBN$, or $\triangle XCQ$ is equal to the sum of areas of the other two.

26.2.9.2. A closed 14-angled broken line is drawn along the lines of a piece of mesh paper. No line of the graph contains more than two links of the broken line and no two links can go in succession along one horizontal or vertical line. What is the greatest number of self-intersection points that the broken line can have?

26.2.9.3. We drew all diagonals in a regular decagon. How many nonsimilar triangles is it possible to form from all sides and diagonals of the decagon?

26.2.9.4. A 9×9 table contains all integers from 1 to 81. Prove that there exist two adjacent numbers whose difference is not less than 6. (Cf. Problem 26.2.8.2.)

26.2.9.5. See Problem 26.2.7.5.

Grade 10

26.2.10.1. Prove that the equation $x^n + y^n = z^n$ cannot have integer solutions if $x + y$ is prime and n is an odd number > 1 .

26.2.10.2. We drew a mesh of n horizontal and n vertical straight lines on a sheet of paper. How many distinct closed broken, perhaps, self-intersecting, lines of $2n$ segments each can one draw along the lines of the mesh so that each broken line traverses along all horizontal and vertical lines?

26.2.10.3. In a regular 25-gon we drew vectors from the center to all the vertices. How to select several of these 25 vectors for the sum of the selected vectors to be the longest?

26.2.10.4. Let A', B', C', D', E' be the midpoints of the sides of convex pentagon $ABCDE$. Prove that $2S_{A'B'C'D'E'} \geq S_{ABCDE}$.

26.2.10.5*. Consider the sequence $a_1 = a_2 = 1, a_n = \frac{a_{n-1}^2 + 2}{a_{n-2}}$ for $n \geq 3$. Prove that the a_n are integers.

Grade 11

26.2.11.1. Prove that there are no distinct positive integers x, y, z, t such that

$$x^x + y^y = z^z + t^t.$$

26.2.11.2. Prove that of 11 arbitrary infinite decimal fractions one can select two fractions with the difference between them having either an infinite number of zeros or an infinite number of nines in the decimal expression.

26.2.11.3. Find all polynomials $P(x)$ satisfying the identity

$$xP(x-1) = (x-26)P(x) \quad \text{for all } x.$$

26.2.11.4. See Problem 26.2.10.4.

26.2.11.5*. Prove that on a sphere it is impossible to arrange three arcs of great circles of measure 300° each so that no two of them have any common points (endpoints included). (See Fig. 54.)

FIGURE 54. (Probl. 26.2.11.5)

Olympiad 27 (1964)

Tour 27.1

Grade 7

27.1.7.1. In $\triangle ABC$, the heights dropped to sides AB and BC are not shorter than the respective sides. Find the angles of the triangle.

27.1.7.2. On a given circle, there are selected two diametrically opposite points A and B and a third point, C . The tangent to this circle at B meets line AC at M . Prove that the tangent drawn to this circle at C divides BM in halves.

27.1.7.3. Prove that the sum of the digits in the decimal expression of a perfect square cannot be equal to 5.

27.1.7.4. We drew 11 horizontal and 11 vertical intersecting straight lines on a sheet of paper. We call a segment of one of the straight line drawn that connects two neighboring intersections a "link". What least number of links must we erase in order for each intersection to be a junction of not more than 3 links?

27.1.7.5. Consider the sequence $a_0 = a_1 = 1$; $a_{n+1} = a_{n-1}a_n + 1$ for $n = 1, 2, \dots$. Prove that a_{1964} is not divisible by 4.

Grade 8

27.1.8.1. See Problem 27.1.7.1.

27.1.8.2. Find all positive integers n such that $(n - 1)!$ is not divisible by n^2 .

27.1.8.3. Solve in integers for unknowns x , y and z :

$$\sqrt{x + \sqrt{x + \dots + \sqrt{x}}} = z \quad (y\text{-many square roots}).$$

27.1.8.4. See Problem 27.1.9.4 a) below.

27.1.8.5. Take the sums of digits of all numbers from 1 to 1 000 000. Next, take the sums of digits of the numbers obtained, etc., until you get 1 000 000 one-digit numbers. Which number is more numerous among them: 1 or 2?

Grade 9

27.1.9.1. Solve the system in positive numbers:

$$\begin{cases} x^y = z, \\ y^z = x, \\ z^x = y. \end{cases}$$

27.1.9.2. Prove that the product of two consecutive positive integers is not a power of any integer.

27.1.9.3. Given that $a - k^3 \vdots 27 - k$ for any integer k , except $k = 27$, find a .

27.1.9.4. a) Prove that if all angles of a hexagon are equal, then its sides satisfy the following relations:

$$a_1 - a_4 = a_5 - a_2 = a_3 - a_6.$$

b) Prove that if the lengths of segments a_1, \dots, a_6 satisfy the above relations, then one can construct from them an equiangular hexagon.

27.1.9.5. In quadrilateral $ABCD$, We drop perpendiculars from vertices A and C to diagonal BD , and from vertices B and D , to AC . Let M, N, P, Q be the bases of the perpendiculars. Prove that quadrilaterals $ABCD$ and $MNPQ$ are similar. (See Fig. 55.)

FIGURE 55. (Probl. 27.1.9.5)

FIGURE 56. (Probl. 27.2.7.1)

Grades 10 – 11

27.1.10-11.1. A number N is a perfect square and does not end with a zero. After erasing its two last digits, one gets another perfect square. Find the greatest N with this property.

27.1.10-11.2. See Problem 27.1.8.3.

27.1.10-11.3. It is known that for any integer $k \neq 27$ the number $a - k^{1964}$ is divisible by $27 - k$. Find a . (Cf. Problem 27.1.9.3.)

27.1.10-11.4. See Problem 27.1.8.4.

27.1.10-11.5. What is the least number of nonintersecting tetrahedrons into which a cube can be divided?

Tour 27.2

Grade 7

27.2.7.1. We select an arbitrary point B on segment AC . Segments AB, BC , and AC are diameters of circles T_1, T_2 and T_3 , respectively. Consider a straight line through B ; let it intersect T_3 at P and Q , and let it intersect T_1 and T_2 at R and S , respectively. Prove that $PR = QS$. (See Fig. 56.)

27.2.7.2. $2n$ persons attended a party. Everyone was acquainted with at least n guests. Prove that it is possible to select 4 of the guests and seat them at a round table so that each sits next to his or her acquaintances.

27.2.7.3. 102 points, no three of which are on the same straight line, are chosen in a square with side 1. Prove that there exists a triangle with vertices at these points and of area less than $\frac{1}{100}$.

27.2.7.4. Through opposite vertices A and C of quadrilateral $ABCD$ a circle is drawn intersecting AB, BC, CD and AD at M, N, P and Q , respectively. Suppose $DP = DQ = BM = BN = R$, where R is the radius of the circle. Prove that $\angle ABC + \angle ADC = 120^\circ$.

27.2.7.5. For what positive integers a the equation $x^2 + y^2 = axy$ has a solution for x and y in positive integers?

Grade 8

27.2.8.1. Each of n glasses of sufficient capacity contains the same amount of water as the other glasses do. At one step we may pour as much water from any glass into any other as the receiving glass already contains. For what n is it possible to empty all glasses into one glass in a finite number of steps?

27.2.8.2. Consider three points A, B, C on the same straight line and one point, O , not on it. Denote by O_1, O_2, O_3 the centers of circles circumscribed around triangles $\triangle OAB, \triangle OBC, \triangle OAC$. Prove that the points O_1, O_2, O_3 and O are all on one circle, see Fig. 57.

FIGURE 57. (Probl. 27.2.8.2)

27.2.8.3. Two players sit at a 99×99 tic-tac-toe board. The first player draws a “ \times ” in the central square. Then the second player may draw a “ O ” in any of the eight squares adjacent to the \times . Now, the first player draws a \times in any of the squares adjacent to those already occupied, and so on. The first player wins if (s)he can draw his/her \times in any corner square. Prove that the first player can always win.

27.2.8.4. Inside an equilateral (not necessarily regular) heptagon $A_1A_2A_3A_4A_5A_6A_7$ an arbitrary point O is chosen. Denote by $H_1, H_2, H_3, H_4, H_5, H_6, H_7$ the bases of the perpendiculars dropped from O to $A_1A_2, A_2A_3, A_3A_4, A_4A_5, A_5A_6, A_6A_7$, respectively. It is known that points $H_1, H_2, H_3, H_4, H_5, H_6, H_7$ belong to the sides themselves, not to their extensions. Prove that

$$\begin{aligned} A_1H_1 + A_2H_2 + A_3H_3 + A_4H_4 + A_5H_5 + A_6H_6 + A_7H_7 \\ = H_1A_2 + H_2A_3 + H_3A_4 + H_4A_5 + H_5A_6 + H_6A_7 + H_7A_1. \end{aligned}$$

27.2.8.5*. 101 distinct points are chosen at random in a square of side 1 (not necessarily inside it, some points might lie on the sides), so that no three of the points belong to one straight line. Prove that there is a triangle with vertices at some of the fixed points whose area does not exceed 0.01.

Grade 9

27.2.9.1. See Problem 27.2.8.1.

27.2.9.2. See Problem 27.2.8.4.

27.2.9.3. Prove that any non-negative even number $2n$ can be uniquely represented in the form $2n = (x + y)^2 + 3x + y$, where x and y are nonnegative integers.

27.2.9.4. In $\triangle ABC$, side BC is equal to a halfsum of the other two sides ($AB \neq AC$). Prove that the bisector of $\angle BAC$ is perpendicular to the segment connecting the centers of the inscribed and circumscribed circles.

27.2.9.5*. On a graph paper consider a closed broken line whose vertices are in the nodes of the grid and all segments of the broken line are equal. Prove that the number of the segments of such a broken line is even.

Grade 10

27.2.10.1. n beakers contain n distinct liquids, there is also an empty beaker. We assume that each beaker is continuously graded so that we can measure the volume of liquid inside it. Is it possible to compose uniform mixtures in each beaker inside a finite length of time? In other words, is it possible to arrange so that each of the n beakers contains exactly $\frac{1}{n}$ of the initial quantity of each liquid and one beaker is empty?

27.2.10.2. We have a system of n points on a plane such that for any two points there is a movement of the plane sending the first point to the second one and the whole system into itself. Prove that all points of such a system belong to a circle.

27.2.10.3. In $\triangle ABC$, side BC is equal to a halfsum of the other two sides. Prove that vertex A , the midpoints of AB and AC and the centers of the inscribed and circumscribed circles belong to one circle.

27.2.10.4. See Problem 27.2.9.5.

27.2.10.5*. Several positive integers are written on each of infinitely many cards so that for any n there is exactly n cards on which the divisors of n are written. Prove that every positive integer is encountered on at least one card.

Grade 11

27.2.11.1. Several vectors begin from point O on a plane; the sum of their lengths is equal to 4. Prove that it is possible to select several of these vectors (perhaps, one) the length of whose sum (whose length) is greater than 1.

27.2.11.2. See Problem 27.2.8.3.

27.2.11.3. In $\triangle ABC$, sides AB and AC are of different length, side BC is equal to their halfsum.¹ Consider the circle through A and the midpoints of AB and AC . Consider the tangents to the circle pass through the triangle's center of mass. Prove that a) one of the tangent points is the center of the circle inscribed in $\triangle ABC$, b) the straight line through the intersection point of medians and the intersection point of bisectors of $\triangle ABC$ is tangent to this circle. (Cf. Problem 27.2.8.4 and 27.2.10.3.)

27.2.11.4*. A pie is of the form of a regular n -gon inscribed in a circle of radius 1. One straight cut of length 1 is made from the midpoint of each side. Prove that in this way we always cut off a piece of the pie (even if we'd rather not).

27.2.11.5*. Once upon a time there were $2n$ knights at King Arthur's court; each of the knights had not more than $n - 1$ enemies among the knights present. Prove that Merlin, King Arthur's counsellor, can place the knights at the Round Table so that no knight will have his enemy as a neighbor.

Olympiad 28 (1965)

Tour 28.1

Grade 8

28.1.8.1. Given circle S , straight line a intersecting S , and a point M . Draw a line b through M so that the part of b inside S is bisected by a . (See Fig. 58.)

FIGURE 58. (Probl. 28.1.8.1)

28.1.8.2. Prove the validity of the following test of divisibility by 37. *Divide the decimal expresion of n into groups of 3 digits from right to left. If the sum of the resulting three-digit numbers is divisible by 37, then $n \div 37$.* (These three-digit numbers may begin with zeros and, therefore, be actually two-digit or one-digit numbers; e.g., the left-most group can be so.)

28.1.8.3. Given straight line a and two nonparallel segments AB and CD on one side of it. Find a point M on a such that $\triangle ABC$ and $\triangle CDM$ have equal areas.

28.1.8.4. 30 teams participate in a soccer tournament. Prove that during the tournament there always exist two teams which have played the same number of games.

Grade 9

28.1.9.1. A six-digit number is divisible by 37. All its digits are different. Prove that one can form from the same digits another six-digit number divisible by 37.

28.1.9.2. Inside a given triangle ABC , find a point O such that the ratio of areas of triangles $\triangle AOB$, $\triangle BOC$ and $\triangle COA$ is equal to $1 : 2 : 3$.

¹The original formulation was vague. It did not state that $AB \neq BC$. But if $AB = BC$ the circle's center of mass lies on the circle causing a degeneracy. It was also unclear whether the center of mass will automatically be outside the circle constructed, or to have it outside is an extra condition.

28.1.9.3. Consider $\triangle ABC$ with $AB > BC$ and bisectors AK and CM , where K is on BC and M on AB . Prove that $AM > MK > KC$.

28.1.9.4. In Illiria, some pairs of towns are connected by direct airlines. Prove that there exist two towns in Illiria that are connected with the same number of other towns. (Cf. Problem 28.1.8.4.)

28.1.9.5. An elderly woman decides to reduce noise from the flat below by placing along her (rectangular) corridor rectangular mats of the same width as the corridor. The mats cover the entire floor and even overlap so that certain portions of the floor are covered by several layers.

Prove that it is always possible to remove several mats, perhaps taking them from underneath and leaving the others in their original positions, so that the floor will remain completely covered and the combined length of the remaining mats will be less than twice the length of the corridor.

Grade 10

28.1.10.1. The circles O_1 and O_2 are inside $\triangle ABC$. They are tangent to each other externally; moreover, O_1 is tangent to AB and BC , and O_2 is tangent to AB and AC . Prove that the sum of the radii of these circles is greater than the radius of the circle inscribed in ABC . (See Fig. 59.)

FIGURE 59. (Probl. 28.1.10.1)

FIGURE 60. (Probl. 28.1.10.3)

28.1.10.2. See Problem 28.1.9.1.

28.1.10.3. The endpoints of a segment of fixed length slide along the legs of a given angle. The perpendicular to the segment is erected from its midpoint. Prove that the distance from the base of the sliding perpendicular to the point where it meets the bisector of the angle is a constant. (See Fig. 60.)

28.1.10.4. Let $x > 2$. Somebody writes on cards the numbers $1, x, x^2, x^3, \dots, x^k$ (a number per card). Then Somebody puts some of the cards in her right pocket, some in her left pocket, and throws away the rest.

Prove that the sum of the numbers in Somebody's right pocket cannot be equal to the sum of the numbers in her left pocket. (Cf. Problem 28.1.11.1.)

28.1.10.5. A paper square has 1965 perforations. No three of the 1969 points — the union of the perforation points with the square's vertices — lie on the same straight line. We cut along several nonintersecting line segments with endpoints at perforations or vertices on the square. It turns out that the cuts divide the square into triangles inside which there are no perforations. How many cuts were made and how many triangles were obtained?

Grade 11

28.1.11.1. Each coefficient of a polynomial $f(x)$ is equal to 1, 0 or -1 . Prove that all real roots (if any) of the polynomial lie on the segment $[-2, 2]$.

28.1.11.2. Given three points on a plane, construct three circles tangent to one another at these points. Consider all possible cases.

28.1.11.3. In the quadratic equation $x^2 + px + q = 0$, the coefficients p and q independently take on all values from segment $[-1, 1]$. Find the set of real roots of these quadratic equations.

28.1.11.4. Given circle O , a point A on it, the perpendicular erected at A to the plane on which O lies, and a point B on this perpendicular. Find the locus of bases of perpendiculars drawn from A to the straight line through B and any point on circle O .

28.1.11.5*. Given 20 cards with each of the figures $0, \dots, 9$ written on two of these cards. Find whether it is possible to arrange the cards in a row so that the zeros are next to one another, the 1's have one card between them, the twos have two cards between them, etc., and the nines have nine cards between them.

Tour 28.2

Grade 8

28.2.8.1*. Given an infinite in both ways sequence

$$\dots, a_{-n}, \dots, a_{-1}, a_0, a_1, \dots, a_n, \dots, \quad \text{where } a_n = \frac{1}{4}(a_{n-1} + a_{n+1}),$$

prove that if some two of its terms (not necessarily adjacent) are equal, then the sequence contains an infinite number of pairs of equal terms.

28.2.8.2. We place a rectangular billiard table of size 26×1965 so that two of its longer sides are oriented North-South and its shorter sides are oriented East-West. The pockets are only at the vertices of the rectangle. A ball is shot from the lower left (SW) pocket of the billiard at an angle of 45° . Prove that after several rebounds from the sides the ball will reach the upper left (NW) pocket.

28.2.8.3. We divide two paper discs of different size into 1965 equal sectors. On each of the discs we select at random 200 sectors and paint them red. We put the smaller disc on top of the bigger one, so that their centers coincide and the sectors of one lie just over the sectors of the other. A *position* is the new relation between discs that they assume after we rotate the smaller disc through all angles that are multiples of $\frac{2\pi}{1965}$, while the bigger disc is fixed. Prove that in at least 60 positions not more than 20 red sectors of both discs coincide.

28.2.8.4*. In a fairyland, a row of houses, with square foundations of side a , stands between two parallel streets. The distance between the streets is $3a$, and the distance between two neighboring houses is $2a$. One street is patrolled by cops who stroll at a distance of $9a$ from one another, at a constant speed v no matter what. When the first cop passes the middle of a certain house, a robber appears, exactly opposite the cop, on the other street, see Fig. 61.

FIGURE 61. (Probl. 28.2.8.4)

The robber is doomed to move with a constant speed; thanks to a Good Fairy the robber can reach any value of speed, without any acceleration, instantaneously. At what constant speed and in which direction should the robber move along that street so that no cop spots him?

Grade 9

28.2.9.1. See Problem 28.2.10.1.

28.2.9.2. We shot a ball from a vertex of a rectangular billiard table with pockets at its vertices at an angle of 45° to the side. If the ball reaches a pocket, it falls into it. After a while the ball reached the midpoint of a certain side. Prove that it could not have already touched the midpoint of the opposite side.

28.2.9.3. See Problem 28.2.8.1.

28.2.9.4. See Problem 28.2.10.2.

28.2.9.5. Find the locus of the centers of equilateral triangles circumscribed around an arbitrary given triangle.

Grade 10

28.2.10.1. We have 11 sacks of coins and a balance with two pans and a hand dial that indicates which pan contains a heavier load and what is the difference in their weights. We can weigh any number of coins from any sack. We know that all coins in one sack are counterfeit, and all other coins are genuine. All genuine coins are of weight x , whereas all counterfeit coins are of weight y , where neither x nor y are known. What is the least number of weighings needed to determine which sack has counterfeit coins?

28.2.10.2. On a $n \times n$ piece of graph paper, we arrange black and white cubes so that each cube stands on exactly one 1×1 square formed by the paper's mesh. We had formed the first layer of n^2 cubes when The Rule was issued: *two cubes are called neighboring to each other if they have a common face; each black cube must have an even number of neighboring white cubes, and each white cube must have an odd number of neighboring black cubes.*

So we arranged the second layer of cubes in such a way that all cubes of the first layer obeyed The Rule. If all cubes of the second layer also satisfy The Rule, we are done. If this is not the case, we have to fill in the third layer so that all cubes of the second layer satisfy The Rule, etc. Does there exist an arrangement of cubes in the first layer for which this process is infinite?

28.2.10.3. Let p and q be odd integers. A $p \times 2q$ rectangular billiard table has pockets at each vertex and in the midpoints of sides of length $2q$. A ball is shot from a vertex at an angle of 45° to the sides. Prove that the ball will wind up in one of the middle pockets. (Cf. Problem 28.2.9.2.)

28.2.10.4. All integers 1 to $2n$ are written in a row in an arbitrary order. Then to each integer the number of its place in the row is added. Prove that among the sums obtained there are at least two that have the same remainders after division by $2n$.

28.2.10.5*. In a box there are two smaller boxes, in each of which there are two more boxes, etc. There are 2^n smallest boxes, each contains a coin. Some of these coins are heads up, some tails up. In one move, any box may be turned upside down, together with everything it contains. Prove that in not more than n moves the boxes may be so arranged that the number of coins with heads up is equal to the number of coins with tails up.

Grade 11

28.2.11.1. Find all primes of the form $p^p + 1$ and of not more than 19 decimal digits, where p is a positive integer.

28.2.11.2. Prove that the last digits of numbers of the form n^n , where n is a positive integer, constitute a periodic sequence.

28.2.11.3*. Given plane P and two points A and B on either side of it, construct a sphere through these points that cuts in P a disc of the smallest possible area.

28.2.11.4. Consider a non-convex and non-selfintersecting polygon on a plane. Let D be the union of points on those diagonals of the polygon that do not go outside its limits (i.e., are either entirely inside it or partly inside and partly on its boundary; the endpoints of these diagonals should also belong to D). Prove that any two points of D may be connected by a broken line contained entirely within D .

28.2.11.5. Each square of an $M \times M$ table contains nonnegative integers so that if a 0 is at the intersection of a row and a column, then the sum of the numbers in this row and this column is not less than M . Prove that the sum of all numbers in the table is not less than $\frac{M^2}{2}$.

Olympiad 29 (1966)

Tour 29.1

Grade 8

29.1.8.1. Find the locus of the centers of all rectangles inscribed in a given $\triangle ABC$ with one side of the rectangles on AB .

29.1.8.2. Find all two-digit numbers that being multiplied by an integer yield a product whose penultimate digit is 5.

29.1.8.3. See Problem 29.1.9-11.1.

29.1.8.4. See Problem 29.1.9-11.5.

29.1.8.5*. From a complete set of 28 dominoes, we remove all dominoes that have 6 dots (on any half, not in sum). Is it possible to arrange the remaining dominoes in a chain?

Grades 9 – 11

29.1.9-11.1. Solve in positive integers the system of equations

$$\begin{cases} x + y = zt, \\ z + t = xy. \end{cases}$$

29.1.9-11.2. For what value of k is the expression $A_k = \frac{19^k + 66^k}{k!}$ the greatest?

29.1.9-11.3. We place a convex pentagon inside a circle, so that its vertices are either on the circle or inside it. Prove that at least one of the pentagon's sides is not longer than the side of a regular pentagon inscribed in this circle.

29.1.9-11.4. Prove that the positive integers k , for which $k^k + 1$ is divisible by 30, constitute an arithmetic progression and describe that progression.

29.1.9-11.5. In checkers, what is the greatest number of kings that may be arranged on the black squares of an 8×8 checker-board, so that each king may be jumped by at least one other king?

Tour 29.2

Grade 8

29.2.8.1. Divide a line segment into six equal parts with a ruler and compass constructing not more than eight curves (straight lines or arcs).

29.2.8.2*. Let $a_1 = 1$ and for $k > 1$ define $a_k = [\sqrt{a_1 + a_2 + \cdots + a_{k-1}}]$, where $[x]$ denotes the integer part of x . Find a_{1000} .

29.2.8.3*. There is a test that for any set of balls can determine whether the set contains any radioactive balls, but it cannot tell how many of the balls are radioactive. We know that two of the given 19 balls are radioactive. Find both the radioactive balls after 8 tests. (Cf. Problem 29.2.9-11.3 below).

29.2.8.4. A subway system has not more than four stations along each line, not more than three of which are intersections with the other lines. Moreover, not more than two lines meet at any of the intersections. What greatest number of lines can such a system have if it is possible to get from any station to any other station with not more than two train changes?

29.2.8.5*. Prove that there exists k such that the first 4 digits of $k!$ are 1966.

Grades 9 – 11

29.2.9-11.1. See Problem 29.2.8.1.

29.2.9-11.2*. See Problem 29.2.8.2, where $a_1 = 1966$, and find a_{1966} .

29.2.9-11.3*. There is a test that for any set of balls can determine whether the set contains any radioactive balls, but it cannot tell how many of the balls are radioactive. We know that two of the given 11 balls are radioactive. Prove that fewer than seven tests do not guarantee the discovery of both radioactive balls, whereas one can determine them by seven tests.

29.2.9-11.4. Given a collection of weights 1, 2, ..., 26 g. Select 6 weights so that it is impossible to compose with the help of (some or all of) these 6 weights two piles of equal weight. Prove that it is impossible to select 7 weights with the same property.

29.2.9-11.5. On an 11×11 checker-board 22 squares are marked so that exactly two of the marked squares lie in each column and each row. Two arrangements of marked squares are considered *equivalent* if in any number of permutations of the columns and/or (independent) permutations of rows one arrangement can be obtained from the other. How many nonequivalent arrangements of marked squares are there?

Olympiad 30 (1967)

Tour 30.1

Grade 8

30.1.8.1. Do there exist two consecutive positive integers such that the sum of the digits of each of them is divisible by 125? Either find the smallest such pair of numbers or prove that they do not exist.

30.1.8.2. Given $\triangle ABC$, find a point M on side AB or its extension so that the sum of the radii of the circles circumscribed about $\triangle ACM$ and $\triangle BCM$ is minimal.

30.1.8.3. A spy must cipher his (her) message. For this (s)he wants to divide all decimal “words” — sets of ten signs, each either a dot or a dash — into two groups, so that any two words of the same group differ in not less than three places. Either describe such a division or prove that the spy’s assignment is hopeless.

30.1.8.4. Given $\triangle ABC$, find the locus of all points M for which $\triangle ABM$ and $\triangle BCM$ are isosceles.

30.1.8.5. In the city of Fuchs, Ostap Bender organized a distribution of elephants to people. There were present 28 trades union members and 37 non-members. Ostap distributed the elephants so that the shares of all trades union members were equal, and the shares of non-members were also equal. It turned out that there existed only one such distribution (of all elephants). What greatest number of elephants could O. Bender have had?

Remark. O. Bender is a main character of a satiric dilogy by I. Il’f and E. Petrov. It became immensely popular since it had been first published in the late ’20s. A part of it, translated into English in the ’30s under the title *The Little Golden Calf*, is a series of adventures of Jeff Peters and Andy Tuckers type. Among O. Bender’s rackets were paid popular lectures and prophesies. An announcement written in the spirit of ‘Royal Nonsuch’ from Huck Finn’s adventures said that after the lecture O. Bender was to distribute elephants. At the time of Bender’s adventures the majority of the audience — hicks and red-necks — had no hope to get ‘elephants’; all of them, even the “privileged” trades union members, were primarily interested not even in future-telling but in the immediate *now* and prone to ask Bender “Why there is no butter on sale?” or “Are you a Jew?” (Miraculously, this book is even more timely now, in late ’90s.)

Grade 9

30.1.9.1. A maze consists of n circles, all tangent to straight line L at M . All circles are on the same side of L and their lengths form a geometric progression with denominator 2. Two pedestrians enter the maze at different moments. Their speeds are equal but the directions of their trajectories are different. Each of them circumvents all circles, starting with the smaller, in increasing order and, having circumvent the greatest, enters the smallest one again. Prove that the pedestrians will meet each other. (See Fig. 62.)

FIGURE 62. (Probl. 30.1.9.1)

FIGURE 63. (Probl. 30.1.9.2)

30.1.9.2. Is it possible to cut a square pie into 9 pieces of equal area by choosing two points inside the square and connecting each of them by straight cuts with all vertices of the square? If it is possible, how can the two points be found? (See Fig. 63.)

30.1.9.3. See Problem 30.1.8.2.

30.1.9.4. Consider integers with the sum of their digits divisible by 7. What is the greatest difference between two consecutive such integers?

30.1.9.5. We transpose the first 12 digits of a 120-digit number in all possible ways, and out of the 120-digit numbers obtained we randomly choose 120 numbers. Prove that the sum of the chosen numbers is divisible by 120.

Grade 10

30.1.10.1. Inside a square consider k points ($k > 2$). Into what least number of triangles must we divide the square for each triangle to contain not more than one point?

30.1.10.2. Prove that in a circle of radius 1 there may be not more than 5 points such that the distance between any two of them is greater than 1.

30.1.10.3. Prove that the equation $19x^3 - 17y^3 = 50$ has no integer solutions.

30.1.10.4. An infinite pie occupying all space has raisins of diameter 0.1 with centers at the points with integer coordinates. Finitely many planes cut the pie. Prove that there still exists an uncut raisin.

30.1.10.5. Of the first k primes $2, 3, 5, \dots, p_k$ ($k > 4$) we compose all possible products every prime entering the product not more than once, e.g. $3 \cdot 5, 3 \cdot 5 \cdot \dots \cdot p_k, 11 \cdot 13, 7$, etc. Let S be the sum of all such products. Prove that $S + 1$ is the product of more than $2k$ prime factors.

Tour 30.2

Grade 7

30.2.7.1. In $\triangle ABC$, consider heights AE, BM and CP . It turns out that $EM \parallel AB$ and $EP \parallel AC$. Prove that $MP \parallel BC$.

30.2.7.2. Four electric bulbs must be installed above a square skating-rink in order to illuminate it completely. At what least height may the lamps be hung, if each lamp illuminates a disc of a radius equal to the lamps' height from the floor?

30.2.7.3. Prove that there exists an integer q such that the decimal expression of $q \cdot 2^{1000}$ contains no zeros. Cf. Problem 30.2.8.1.

30.2.7.4. A number y is obtained from a positive integer x by a permutation of its digits. Prove that $x + y \neq 99\dots99$ (1967 nines).

30.2.7.5. Spotlights, each of which illuminates a right angle, are placed at four given points on a plane. The sides of the illuminated angles may be directed only to the north, south, west or east. Prove that the spotlights may be so directed that they illuminate the entire plane; see Fig. 64.

FIGURE 64. (Probl. 30.2.7.5)

Grade 8

30.2.8.1. See Problem 30.2.7.3 for $q \cdot 2^{1967}$.

30.2.8.2. Denote by $d(N)$ the number of divisors of N (1 and N are also considered as divisors of N). Find all N such that $N/d(N) = p$ is a prime.

30.2.8.3. A square is constructed on each side of a right triangle and the entire figure is inscribed in a circle. For what right triangles is this possible? Cf. Problem 30.2.9.3.

30.2.8.4. A black king and 499 white rooks stand on a 1000×1000 chess-board. Black and white pieces move in turn. Prove that whatever the strategy of the whites, the king may always commit suicide after several moves (i.e., get in the way of a rook).

30.2.8.5. Seven children decided to visit seven movie theaters one day. At each movie theater the shows started at 9.00, 10.40, 12.20, 14.00, 15.40, 17.20, 19.00 and 20.40 (altogether 8 shows). Six children went together to each show, and each time the seventh kid (not necessarily the same person each time) decided to be independent and went to another movie theater. By night each kid had been to each of the seven theaters chosen. Prove that there was a show in each movie that none of the children saw.

Grade 9

30.2.9.1. A number y is obtained from a positive integer x by a permutation of its digits and $x + y = \underbrace{100\dots00}_{200 \text{ zeros}}$. Prove that x is divisible by 50.

30.2.9.2. Given a sequence of positive integers x_1, x_2, \dots, x_n , each not greater than M , and with $x_k = |x_{k-1} - x_{k-2}|$ for all $k > 2$. Determine the greatest possible length of this sequence.

30.2.9.3. We construct a square outwards on each side of a right $\triangle ABC$. It turns out that all vertices of the squares distinct from A, B, C lie on one circle. Prove that $\triangle ABC$ is isosceles. Cf. Problem 30.2.8.3.

30.2.9.4*. Let \overleftarrow{N} be the number N written in reverse order, e.g., $\overleftarrow{1967} = 7691$, $\overleftarrow{450} = 54$. For any positive integer N divisible by K the number \overleftarrow{N} is also divisible by K . Prove that K is a divisor of 99.

30.2.9.5. A king of Spain decided to rearrange the portraits of his predecessors that hung in a circular tower of his castle. He ruled, however, that only two adjacent portraits be interchanged in one day and that, moreover, they could not be the portraits of the kings one of whom immediately succeeded the other. Two distinct arrangements that could have been obtained from each other, if the castle could rotate, were ordered to be considered as *identical*. Prove that, following this Rule, the king can always find any new arrangement of the portraits regardless of their initial positions.

Grade 10

30.2.10.1*. Let m and $n - k$ be relatively prime given numbers. We are given an $n \times n$ table filled in with numbers as follows: numbers $1, 2, \dots, n$ are written in the first row; if some row contains the numbers

$$a_1, \dots, a_k, a_{k+1}, \dots, a_m, a_{m+1}, \dots, a_n$$

then the next row contains the same numbers but in the following order:

$$a_{m+1}, \dots, a_n, a_{k+1}, \dots, a_m, a_1, \dots, a_k.$$

Prove that, after the table is filled, each column contains all numbers 1 to n .

30.2.10.2. See Problem 30.2.9.3.

30.2.10.3. Is it possible to arrange the numbers $1, 2, \dots, 12$ on a circle so that the difference between any two adjacent numbers is 3, 4 or 5?

30.2.10.4. Eight spotlights are situated at eight given points in space, each spotlight illuminates a trihedral angle with mutually perpendicular faces. Prove that the spotlights may be turned so as to illuminate the entire space. (Cf. Problem 30.2.7.5.)

30.2.10.5. Consider all possible n -digit numbers, $n \geq 2$, composed of figures 1, 2 and 3. At the end of each of these n -digit numbers we write a 1, 2 or 3 in such a way that if two numbers differ in all the corresponding digits, then we write additional different digits at their ends (one digit each). Prove that there exists an n -digit number which contains only one 1 and at whose end a 1 is written.

Olympiad 31 (1968)

Tour 31.1

Grade 7

31.1.7.1. Number 4 has the following property: when divided by q^2 , for any q , the remainder is less than $\frac{q^2}{2}$. Find all numbers with the same property.

31.1.7.2. Arrange 16 numbers in a 4×4 table so that their sum along any vertical, horizontal or diagonal line is equal to zero. We assume that the table has 14 diagonals altogether.

31.1.7.3. Prove that for any three given numbers, each $< 1\,000\,000$, there is a number < 100 that is relatively prime to every one of the given numbers.

31.1.7.4. How may 50 cities be connected by the least possible number of airlines so that from any city one could get to any other by changing airplanes not more than once (i.e., using two planes)?

Grade 8

31.1.8.1. 12 people took part in a chess tournament. After the end of the tournament every participant made 12 lists. The *first list* consisted of the author; the *second list* – of the author and of those (s)he has beaten; and so on; the *12-th list* consisted of all the people on the 11-th list and those they have beaten. It is known that the 12-th list of every participant contains a person who is not on the participant's 11-th list. How many games ended in a draw?

31.1.8.2. Given numbers 4, 14, 24, \dots , 94, 104, prove that it is impossible to strike out first one number, then another two, then another three, and then another four, so that after each striking out the sum of the remaining numbers is divisible by 11.

31.1.8.3. Is it possible to inscribe a convex heptagon $A_1A_2A_3A_4A_5A_6A_7$ with angles $\angle A_1 = 140^\circ$, $\angle A_2 = 120^\circ$, $\angle A_3 = 130^\circ$, $\angle A_4 = 120^\circ$, $\angle A_5 = 130^\circ$, $\angle A_6 = 110^\circ$, $\angle A_7 = 150^\circ$ in a circle?

31.1.8.4. Find 100 numbers such that

$$\begin{aligned} x_1 &= 1; \\ 0 &\leq x_2 \leq 2x_1; \\ 0 &\leq x_3 \leq 2x_2; \\ &\dots\dots\dots \\ 0 &\leq x_{99} \leq 2x_{98}; \\ 0 &\leq x_{100} \leq 2x_{99}; \end{aligned}$$

so that the expression $S = x_1 - x_2 + x_3 - x_4 + \dots + x_{99} - x_{100}$ is the greatest possible.

31.1.8.5. Is it possible to arrange 1000 segments on a plane so that the endpoints of every segment are on other segments but not at their endpoints?

Grade 9

31.1.9.1. Is there a quadrilateral $ABCD$ of area 1 such that for any point O inside it the area of at least one of the triangles $\triangle OAB$, $\triangle OBC$, $\triangle OCD$, or $\triangle OAD$ is an irrational number?

31.1.9.2. Cf. Problem 31.1.8.5 for 1968 segments.

31.1.9.3. A corridor 100 meters long is covered with 20 rugs of the same width as the corridor and of a combined length of 1000 meters. What greatest number of uncovered parts may the corridor have?

31.1.9.4. Is it possible to select 100 000 telephone numbers consisting of 6 digits each so that if we simultaneously strike out the k -th digit ($k = 1, 2, 3, 4, 5, 6$) of every number, we get all numbers 00 000 to 99 999?

31.1.9.5. Prove that if p and q are primes and $q = p + 2$, then $p^q + q^p \div p + q$.

Grade 10

31.1.10.1*. 100 airplanes (one in the lead, 99 following) take off simultaneously from the same airport. A plane with a full tank of fuel can cover a distance of 1000 km. During a flight, fuel may be transferred from one plane to another. A plane that gave all its fuel to the other planes makes a gliding landing. How should the flight be organized for the leading plane to fly as far as possible?

31.1.10.2. Two people are playing a game. There are two piles containing 33 and 35 candies. A player eats up one of the piles and divides the second one into two (not necessarily equal) parts. If (s)he cannot divide a pile because it only has one candy, (s)he eats the candy and wins. Moves are made in turn. Who will win the game, the one who starts or the other party, and how should they play to win?

31.1.10.3. The Rule states: integers m and n belong to the same subset if one can be obtained from the other by striking out two of its adjacent identical digits or two identical groups of digits (for example, the numbers 7, 9 339 337, 93 223 393 447, 932 239 447 belong to the same subset). Is it possible to divide the set of all non-negative integers into 1968 subsets, with at least one number in each, so that the Rule is fulfilled?

31.1.10.4. Using a given sequence of positive numbers $q_1, q_2, \dots, q_n, \dots$, a sequence of polynomials is constructed in the following way:

$$f_0(x) = 1, \quad f_1(x) = x \quad \text{and} \quad f_{n+1}(x) = (1 + q_n) \cdot x f_n(x) - q_n f_{n-1}(x) \quad \text{for } n \geq 1.$$

Prove that all real roots of these polynomials belong to $[-1, 1]$.

31.1.10.5. Given 4 lines l_1, l_2, l_3, l_4 in space, each pair of them skew and no three of them parallel to one plane. Draw a plane, P , such that the intersection points A_1, A_2, A_3, A_4 of these lines with P make a parallelogram. How many solutions are there?

Tour 31.2

Grade 7

31.2.7.1. The vertices of a regular 1968-gon are marked on a plane. Two players, in turn, connect two vertices of the polygon by a segment, obeying the following Rule: *two points may not be connected if one of them is already connected to a point, and segments already drawn may not be intersected by others*. The player who may not make a move, according to the Rule, loses. How should one play to win? Who wins if both play optimally?

31.2.7.2. On a plane, there are given three points. We select any two of them, draw the perpendicular through the midpoint of the segment connecting them, and reflect all 3 points through this perpendicular. Then we again select two points among all the points, the original ones and their reflections, and repeat the procedure *ad infinitum*. Prove that there exists a straight line on the plane such that all points obtained in the end lie on one side of it.

31.2.7.3. Two painters paint a long straight fence consisting of 100 parts. They come every other day, alternately, painting a fence around one plot red or green. The first painter is color-blind and mixes up the colors; (s)he remembers what part of the fence (s)he has painted and what color (s)he has used. (S)he can also feel the fresh paint left after the second painter, but can not tell its color. The first painter tries to make the number of places where green borders red the greatest possible. What maximal number of such places can (s)he get, whatever the second painter does?

31.2.7.4. Let x and y be unknown digits. The 200-digit number 89 252 525 ... 2 525 is multiplied by the number $444x18y27$. It turns out that the 53-rd digit from the right of the product is 1, and the 54-th digit is 0. Find x and y .

31.2.7.5*. A cowboy Jimmy bets with his friends that he can shoot through all the four blades of his fan with one bullet. His fan is constructed so that it can not effectively work as a fan but suits Jimmy fine as a target, see Fig. 65:

FIGURE 65. (Probl. 31.2.7.5)

Each of the four blades is a half-disc. The blades sit on a shaft perpendicularly to it; the distances between the planes of the blades are equal. The diameters bounding the half-discs are slanted with respect

to one another. The shaft rotates at the rate of 50 revolutions per second. Jimmy, as a true cowboy, can shoot whenever needed and his bullet may have any (but constant) speed he wants. Prove that Jimmy can win the bet.

Grade 8

31.2.8.1. Let us divide all positive integers into groups so that there is one number in the first group, two numbers in the second, three numbers in the third and so on. Is it possible to do this so that the sum of elements in every group is the 7-th power of an integer?

31.2.8.2*. Two straight lines on a plane meet at an angle of α . A flea sits on one of the lines. Every second it jumps from the line it sits on to the other line. (The intersection point is considered to lie on both lines.) It is known that the length of each jump is equal to 1 and that the flea never returns to the place where it was just before. After a while the flea returns to its initial point. Prove that α has a rational number of degrees; see Fig. 66.

FIGURE 66. (Probl. 31.2.8.2)

FIGURE 67. (Probl. 31.2.8.3)

31.2.8.3*. A round pie is cut by a special cutter that cuts off a fixed sector of the angle measure α , turns this sector upside down, and then inserts back; after that the whole pie is rotated through an angle of β .

Given $\beta < \alpha < 180^\circ$, prove that after a finite number of such operations (the beginning of the first and the second operations are shown on Fig. 67) every point of the pie will return to its initial place.

31.2.8.4. Consider a paper scroll of bus tickets numbered 000 000 to 999 999. The tickets with the sum of the digits in the even places equal to the sum of the digits in the odd places are marked blue. What is the greatest difference between the numbers on two consecutive blue tickets?

31.2.8.5. The land Farra lies on 1 000 000 000 islands. Boats ply the routes between certain islands every day. Boat routes are organized so that one can get to any island from any other island (it could take a few days). The timetable allows a spy and Major Pronin¹ only one passage per day and there is no other way to get from one island to another except via regular boats. The spy never boards a boat on the 13-th of a month, but Major Pronin is not superstitious and, besides, informers always tell Major Pronin where the spy is. According to the Rule Major Pronin *catches* the spy if they are both on the same island. Prove that Major Pronin will catch the spy.

Grade 9

31.2.9.1. Consider a regular pentagon $A_1A_2A_3A_4A_5$ on a plane. Is there a set of points on a plane with the following property: through any point outside the pentagon it is possible to draw a segment whose endpoints belong to the set and it is impossible to do so through points inside the pentagon.

31.2.9.2. We mark point O_1 on a unit circle and, using O_1 as the center, we mark (by means of a compass) we mark point O_2 on the circle (clockwise starting with O_1). Using point O_2 as a new center, we repeat the procedure in the same direction with the same radius; and so on. After we had marked point O_{1968} a circle is cut through each mark so we get 1968 arcs. How many different arc lengths can we thus procure?

¹A notorious hero of Soviet spycatchers.

32.1.7.3. An international soccer tournament took place in Chile in February. The home team “Colo-Colo” won the first place with 8 points. “Dynamo” Moscow was second with 1 point less. A Brazilian team “Corinthians” was the third with 4 points. The fourth was a Yugoslavian team, “Crvena Zvezda”, also with four points. Prove that from these data it is possible to exactly reconstruct how many other teams participated in the tournament and how many points they got.

32.1.7.4. Prove that no power of 2 can end with four identical digits.

32.1.7.5. 1000 regular wooden 100-gons are nailed to the floor. We stretch a rope around the entire system using the nails. Prove that the polygon formed by the rope has more than 99 vertices. (Cf. Problem 32.1.8.2.)

Grade 8

32.1.8.1. See Problem 32.1.7.4.

32.1.8.2. 57 regular wooden 57-gons are nailed to the floor. We stretch a rope around the entire system using the nails. Prove that the polygon formed by the rope has more than 56 vertices. (Cf. Problem 32.1.7.4.)

32.1.8.3. A white rook is chasing a black knight across a 3×1969 chessboard (they move in turns according to common rules). How should the rook play to jump the bishop if the white makes the first move? (Cf. Problem 32.1.7.1.)

32.1.8.4. Given segment AB . Find the locus of points C such that $m_b = h_a$ in $\triangle ABC$ (see Fig. 68).

FIGURE 68. (Probl. 32.1.8.4)

32.1.8.5. Is it possible to write 20 numbers in a row so that the sum of any three consecutive numbers is strictly positive, and the sum of all 20 numbers is strictly negative? (Cf. Problem 32.1.9.3.)

Grade 9

32.1.9.1. Find all positive integers x such that we can subtract the same nonzero digit a from each digit of x (this means that every digit of x is not less than a) and get the number $(x - a)^2$.

32.1.9.2. The Tolpygo Island is of the form of a polygon. There are several countries on the island. Each country is of the form of a triangle and every two countries bordering along (parts of) their sides have an entire side in common, i.e., a vertex of one triangle never lies on the side of another triangle (except at a vertex). Prove that it is possible to paint the map of the island three colors, one color for each country and so that any two bordering countries are painted different colors.

32.1.9.3. Is it possible to write 50 numbers in a row so that the sum of any 17 consecutive numbers is strictly positive, and the sum of any 10 consecutive numbers is strictly negative? (Cf. Problem 32.1.8.5.)

32.1.9.4. See Problem 32.1.7.4.

32.1.9.5. There are 500 towns in Tsar Dodon’s kingdom, each in the form of a regular 37-angled star, with towers at the vertices. Tsar Dodon decides to wall the towers in a convex wall so that every segment of the wall connects two towers. Prove that the wall will consist of not less than 37 segments, provided we count segments on the same straight line only once.

Grade 10

32.1.10.1. Particles emitted by a betatron move along a straight line through two identical thin hoops situated in perpendicular planes so that each hoop passes through the center of the other. Along what straight line should the particles move so as to be as far from the hoops as possible, i.e., so that the shortest distance between the particle and the hoops were the longest possible?

32.1.10.2. An infinite sequence of numbers a_1, \dots, a_n, \dots is periodical, with period 100, i.e., $a_1 = a_{101}$, $a_2 = a_{102}, \dots$. It is known that $a_1 \geq 0$, $a_1 + a_2 \leq 0$, $a_1 + a_2 + a_3 \geq 0$, and, generally, the sums $a_1 + a_2 + \dots + a_n$ are alternately non-negative if n is odd or non-positive if n is even. Prove that $|a_{99}| \geq |a_{100}|$.

32.1.10.3. A pack of cards with their backs down is arranged in a row. If two cards of the same suit are next to one another, or have just one card between them, then the Rule allows us to remove the extreme left one. Besides, the Rule allows us to add any number of cards from other packs to the right hand side of the row. Prove that it is possible to add or remove cards so that in the end only 4 cards are left.

32.1.10.4. Is there a real number h such that $[h \cdot 1969^n]$ is not divisible by $[h \cdot 1969^{n-1}]$ for any positive integer n ?

32.1.10.5. Given square $ABCD$, find the locus of points M such that $\angle AMB = \angle CMD$.

Tour 32.2

Grade 7

32.2.7.1. m and n are two positive integers. All different divisors of m — the numbers a, b, \dots, k — and all different divisors of n — the numbers s, t, \dots, z — are written out (m , and n , and 1 are included). It turns out that

$$a + b + \dots + k = s + t + \dots + z$$

and

$$\frac{1}{a} + \frac{1}{b} + \dots + \frac{1}{k} = \frac{1}{s} + \frac{1}{t} + \dots + \frac{1}{z}.$$

Prove that $m = n$.

32.2.7.2. We strike out two consecutive digits a and b (a preceding b) of the number

$$N = 123456789101112 \dots 9989991000$$

and replace them with the number $a + 2b$; the number a may be an unwritten zero if b is the first digit of N . (Clearly, there are many ways to perform this operation.)

The same operation is repeated with the numbers obtained, and so on. (For example, in one step the numbers 218307, 38307, 117307, 111407, 11837, 118314 may be obtained from 118307.) Prove that several application of this operation turn the given number into 1.

32.2.7.3. A crook acquired a square lot, fenced it in and got permission from the credulous president of his collective farm to perform a few times the following operation: draw a straight line through any two points of the fence, destroy the part of the fence between these two points on one side of the line, and build a new part of the fence symmetric to the destroyed part with respect to the line. Can the crook increase the area of his patch with such manipulations? (See Fig. 69)

FIGURE 69. (Probl. 32.2.7.3)

32.2.7.4. Two players play the following game. Every player, in turn, chooses 9 numbers in the sequence $1, 2, 3, \dots, 100, 101$ and strikes them out. After eleven turns there are two numbers left. The second player then pays the first one the difference between the two numbers in roubles. Prove that the first player can always win at least 55 roubles, no matter how the second one plays.

Note. Students who play this game will be fired from the school.

32.2.7.5. A pearl of radius 3 mm is baked inside a round pudding of radius 10 cm. The Rule allows us to cut the pudding along a straight line with a sharp knife into two (equal or unequal) parts. If the pearl is not found in one cut, (does not occur under the knife), one of the parts may be cut again; if this does not help, it is allowed to cut one of the three obtained parts and so on. Prove that it is possible not to find the pearl after 32 cuts, no matter how they are made. Prove that it is possible to make 33 cuts so that the pearl will be found, no matter where it is.

Grade 8

32.2.8.1. See Problem 32.2.7.2.

32.2.8.2. A white knight is on square $a1$ of a chessboard. Two players take turns daubing one square of the chessboard at a time with bauxite glue. They must do this in such a way that the knight could move according to usual rules onto any clean square without getting stuck. The *loser* is the player who cannot make a move. Who wins provided both play optimally?

32.2.8.3*. Two regular pentagons have one common vertex. The vertices of both pentagons are numbered clockwise 1 to 5, the number of the common vertex being 1. The vertices with the same numbers are connected by straight lines. Prove that these four lines meet at one point. (See Fig. 70)

FIGURE 70. (Probl. 32.2.8.3)

32.2.8.4. Finite sequences of positive integers are composed so that every next number is greater than the square of the preceding one, and the last number of each sequence is equal to 1969 (sequences may have different lengths). Prove that there are fewer than 1969 different such sequences.

32.2.8.5*. 100 cubes are arranged in a row, 77 black and 23 white among them. They are arranged *approximately uniformly*, i.e., if a string of consecutive cubes is marked at one place of the row and the same length is then marked at another place (the strings can intersect), then the number of black cubes in the first string differs from the number of black cubes in the second string by not more than 1; and if the first string begins on the left end of the row, then the number of black cubes in it is *not greater* than that of the second string; and if it terminates the row, then the number of black cubes in it is *not less* than that of the second string. Prove that if another string of 77 black cubes and 23 white cubes satisfies the same conditions, then the white cubes in it occupy the same places as in the first string.

Grade 9

32.2.9.1. Two players play the following game: taking turns, they strike out one number each from the set $\{1, 2, 3, \dots, 27\}$, until there are only two numbers left. If the sum of these numbers is divisible by 5, then the first player wins, otherwise the second one wins. Who wins an optimally played game (the player who begins or the second one)?

32.2.9.2. On a plane, the circle may be traced around a coin. The Rule allows us to use this coin to draw a circle through one or two given points that are sufficiently close to one another. Three points are given on the plane; they can be covered by the coin, and they are not all on one straight line or on the circle equal to the outer circle of the coin. Using the coin construct a fourth point, such that all four points are vertices of a parallelogram.

32.2.9.3*. There are 2^{n-1} different sequences of length n built of 0's and 1's so that for any three sequences there exists a number p such that the p -th digit in all three sequences is 1. Prove that there is exactly one place in each sequence that a 1 occupies.

32.2.9.4*. The time of a new presidential election is approaching in the country Anchuria, where President Miraflores now rules. There are 20 million voters in the country but only 1 percent supports Miraflores (the Anchurian military). Naturally, Miraflores wants to be reelected but he wants the election to look democratic. A “democratic vote”, according to Miraflores, is like this: all voters are divided into equal groups; each of these groups is again divided into some number of equal groups (the groups of voters may be subdivided at distinct stages of the election into distinct numbers of smaller subgroups); then these groups are divided once again, and so on. A representative from each smallest group — an elector — is elected for voting within the greater group; electors of this greater group elect a new elector for voting in the group which is greater than this one, and so on. And, finally, representatives from the greatest groups elect the President. According to the constitution, Miraflores has the right to divide all voters into the groups as he chooses and he instructs his supporters as to how they are to vote. Is it possible for him to organize “democratic” elections so as to be elected?

32.2.9.5. Consider a regular 1000-gon. Its nonintersecting diagonals divide it into triangles. Prove that among these diagonals there are at least 8 of different lengths.

Grade 10

32.2.10.1. Two wizards play the following game. Numbers 0, 1, 2, \dots , 1024 are written out. The first wizard strikes out 512 numbers of (s)he chooses. Then the second wizard strikes out 256 of the remaining numbers. Then the first wizard strikes out 128 of the remaining numbers, and so on. The second wizard strikes out one number during the tenth move; two numbers remain. After that the second wizard pays the first wizard the absolute value of the difference between these numbers in roubles as the fee for the instruction in the exciting game.

How should the first wizard play to gain as much as possible? How should the second wizard play to lose less? How much will the second wizard have to pay the first wizard if both play optimally? (Cf. Problems 32.2.7.4 and 32.2.9.1).

32.2.10.2. A rigid wire is bent to form an equilateral triangle, and its endpoints are soldered. The Rule allows to bend a piece of the wire between any two of its points in such a way that the bent piece is symmetrical to the original one, with respect to the straight line through these two points (if these points coincide, then any line through them will do). This operation may be repeated. Is it possible to obtain a regular hexagon with the same perimeter in several such operations? (Cf. Problem 32.2.7.3, See Fig. 71)

FIGURE 71. (Probl. 32.2.10.2)

32.2.10.3. See Problem 32.2.7.2 with the circle replaced with a sphere of radius 20 cm, and the numbers of cuts — 32 and 33 — replaced with 65 and 66, respectively.

32.2.10.4. Numbers whose sum equals zero are written on the squares of an 8×8 chessboard. Every square is then divided by a vertical and a horizontal line into four square cells. Is it possible to write numbers in these cells so that

- a) there are zeros in all the cells along the sides of the chessboard;
 b) the sums of the numbers in the four cells of each square are equal to the number written in the square before;
 c) the sums of the numbers in the four cells at every vertex of any original square is zero?

32.2.10.5*. Arrange 1969 cubes in a row, some of them (any number between 0 and 1969) white and the rest black, so that the colors are distributed *approximately uniformly* (see Problem 32.2.8.5). Prove that there exist at least 1970 different arrangements of the cubes which meet this requirement.

Olympiad 33 (1970)

Tour 33.1

Grade 7

33.1.7.1. Two black checkers are positioned on two neighboring black squares on the diagonal of an infinite (in two perpendicular directions) chessboard. Is it possible to place several black checkers and a white one on the chessboard so that the white checker could jump all black checkers in one move?

33.1.7.2. We number 99 cards 1 through 99. Then we shuffle the cards, lay them out with the blank sides up and number the blank sides 1 through 99. We sum the two numbers of every card and multiply the 99 sums. Prove that the product is an even number.

33.1.7.3. Point O lies inside an equilateral triangle ABC . It is known that $\angle AOB = 113^\circ$, $\angle BOC = 123^\circ$. Find the angles of the triangle whose sides are equal to segments OA , OB , OC ; see Fig. Probl. 33.1.7.3.

FIGURE 72. (Probl. 33.1.7.3)

33.1.7.4. A set has 100 weights, the difference between every two of them is ≤ 20 g. Prove that it is possible to put the weights on the pans of a balance, 50 weights on each pan, so that the difference between the weights of the pans is ≤ 20 g.

33.1.7.5. There are 1000 cottages in a town X ; just one person occupies each cottage. One day, every man moves to another cottage and every cottage has again one occupant. Prove that it is possible to paint all 1000 cottages blue, green or red so that, for every person, the color of his/her new cottage is distinct from the color of the old cottage.

Grade 8

33.1.8.1. See Problem 33.1.7.2.

33.1.8.2. The circle is inscribed in pentagon $ABCDE$ whose sides are integer numbers and $AB = CD = 1$. Find the length of the segment BK , where K is the tangency point of BC with the circle.

33.1.8.3. There are 16 black points on a rectangular piece of paper. We connect a pair of points by the segment. Consider a rectangle one of whose diagonals is this segment and whose sides are parallel to the sides of the paper. We paint the rectangle red (black points are visible through the red paint). We do so with every pair of points and get a painted figure on the paper.

How many sides can the figure have for various positions of the points?

33.1.8.4. Each pan of a balance has k weights, numbered from 1 to k . The left pan is heavier. It turns out that if we interchange the places of any two weights with the same number, then either the right pan becomes heavier or the two pans reach an equilibrium. For what k this is possible?

33.1.8.5. 12 players took part in a tennis tournament. It is known that every two of them played with one another only once and that there was no player who was always beaten. Prove that among these 12 there are players A , B , and C such that B was beaten by A , C by B , and A by C .

Grade 9

33.1.9.1. 113 kings lived each in his own palace along a straight road. Every morning one of the kings gave a reception which all the others attended, and every evening the servants transported the kings back home. In this way they lived for a year without doing anything else and lieve of absence. Prove that during this year one of the kings who lived at one of the road's ends collected the biggest milage.

33.1.9.2. What is the greatest number of black checkers that one can place on on black squares of an 8×8 checker-board so that a white checker can jump all of them in one move without becoming a king?

33.1.9.3. A given 999-digit number is such that erasing all but any 50 of its successive digits yields a number (that may begin with zeroes or just be zero) divisible by 2^{50} . Prove that the given number is divisible by 2^{999} .

33.1.9.4. Construct triangle $\triangle ABC$ given the radius of the circumscribed circle and the bisector of angle $\angle A$, and knowing that $\angle B - \angle C = 90^\circ$.

33.1.9.5. A wise cockroach who cannot see farther than 1 cm decided to find the Truth. The latter is located at a point D cm away from the cockroach. The cockroach can move step by step, each step not longer than 1 cm, and after each step the cockroach is told whether (s)he is closer to the Truth or not. The cockroach remembers everything, in particular, the directions of his/her steps. Prove that (s)he can find the Truth taking not more than $\frac{3D}{2} + 7$ steps.

Grade 10

33.1.10.1. Given 19 weights, each of an integer mass (in grams) that does not exceed 70 grams, prove that it is impossible to compose more than 1230 different masses of these weights.

33.1.10.2. Two non-intersecting circles O_1 and O_2 are inscribed into angle ABC . Denote by M the tangent point of O_1 and BA , and by P the tangent point of O_2 and BC . Prove that the chords that circles O_1 and O_2 intercept on straight line MP are of equal length. (See Fig. 73)

FIGURE 73. (Probl. 33.1.10.2)

33.1.10.3. We strike out the first digit of the number 2^{1970} and add it to the obtained number. We perform the same operation with the resulting number, and so on, until we get a 10-digit number. Prove that this 10-digit number has two identical digits.

33.1.10.4. Given 200 points on a plane, no three of which are on the same straight line, find whether it is possible to number these points 1 to 200 so that every two of the hundred straight lines through points 1 and 101, 2 and 102, \dots , 100 and 200 intersect.

33.1.10.5. There are crosses in some of the squares of a 100×100 table. It is known that there is at least one cross in every row and in every column. Prove that it is possible to mark 10 rows and 10 columns so that if we erase all crosses in the marked rows and columns, at least one cross will still be left in every unmarked row and column.

Tour 33.2

Grade 7

33.2.7.1. Prove that if a positive integer k is divisible by 999 999 999, then there are more than 8 non-zero digits in its decimal expression.

33.2.7.2. 100 points are marked on a circle of radius 1. Prove that it is possible to find a point on the circle such that the sum of the distances from it to all the other marked points is greater than 100.

33.2.7.3. In a park, 6 narrow alleys of equal length are arranged as the sides and medians of a square. A boy Kolya is running away from his Mother and Father along these alleys. All three can see each other at all times. Can the parents catch the boy if he runs three times faster than any of his parent?

33.2.7.4. A straight cut divides a square piece of paper into two parts. Another straight cut divides one of the parts into two parts. One of the three pieces of paper obtained is again cut into two parts along a line, and so on. What least number of cuts must one do in order to obtain 73 various (perhaps, equal) 30-gons? (Cf. Problem 33.29.4.)

33.2.7.5. King Louis distrusted some of his courtiers. He made a full list of his courtiers and told every one of them to keep an eye on one of the rest. The first one was to spy on the courtier who was spying on the second, the second one was to spy on the one who was spying on the third, and so on, the penultimate one was spying on the courtier who was spying on the last, and the last was spying on the one who was spying on the first. Prove that King Louis had an odd number of courtiers.

Grade 8

33.2.8.1. There are n points inside a circle of radius 1 m. Prove that there exists a point inside the circle or on its perimeter such that the sum of the distances between it and all the other points is not less than n m. (Cf. Problem 33.2.7.2).

33.2.8.2. A monkey ran away from its cage in a small zoo. Two guards are trying to catch it. Both of the guards and the monkey obey The Rule and run only along the paths. There are 6 straight paths in the zoo: 3 long paths form an equilateral triangle, 3 shorter ones connect the midpoints of its sides. Every moment the monkey and the guards can see each other. At the beginning the guards are at one vertex of the triangle and the monkey at another one. Can the guards catch the monkey if the monkey runs three times faster than the guards? (Cf. Problem 33.2.7.3.)

33.2.8.3. In a park grow 10 000 trees. They had been square-cluster¹ planted in 100 rows with 100 trees in each row. What maximum number of these trees can one cut down under the following Rule: standing on any stump, one should be unable to see any other stump behind the trees? The trees are considered to be sufficiently thin.

33.2.8.4. On a roll of paper tape there are written 80 non-zero digits. We cut the tape across into several strips so that there is more than one digit on each strip. Then we add the numbers formed by the digits on each strip. Prove that there exist two distinct ways of cutting the tape to get equal sums.

33.2.8.5. A flat corridor of width 1 m is of the shape of letter Γ and infinite in both directions. There is a flat piece of rigid wire of the form of a nonclosed broken line. Prove that if the distance between the endpoints of the wire is $> 2 + 2\sqrt{2}$ m, then it is impossible to carry the wire along the whole length of the corridor without tilting. Cf. Problem 33.2.9.2.

Grade 9

33.2.9.1. A toy railroad has n components each in the form of a quarter of a circle with radius 10 cm. Their endpoints are joined in succession so that they fit to form a smooth track. Prove that it is impossible to construct a railroad that would begin and end in the same place, with its first and the last components forming an angle of 0, as shown on Fig. 74.

33.2.9.2. A flat infinite L-shaped corridor is of width 1 m. What is the greatest possible distance between the endpoints of a length of rigid wire (not necessarily straight but flat) such that it is possible to pull the wire through the corridor without tilting? (See Fig. 75)

33.2.9.3. There are plus signs in all squares of a 100×100 table. The Rule permits to simultaneously change the signs in all squares of any one row or column. Is it possible to get 1970 minus signs under the Rule?

¹A method of planting advocated by the then ex-leader of the Soviet Union, N. Khrushchev, as the most advanced and best suited to overtake America in agricultural production. The method was abolished. (Perhaps, unwisely?)

FIGURE 74. (Probl. 33.2.9.1)

FIGURE 75. (Probl. 33.2.9.2)

33.2.9.4. A straight line cuts a square piece of paper into two parts. Another straight line cuts one of the parts into two parts. One of the three pieces of paper obtained is again cut into two parts along a line, and so on. What least number of lines must be drawn in order to obtain 100 various (perhaps, (perhaps, identical) 20-gons? Cf. Problem 33.2.7.4.

33.2.9.5. Three spiders and a wingless fly are crawling along the edges of a wire cube. The top speed of the fly is three times that of the spiders. At the beginning, all spiders sat at one vertex of the cube and the fly at the opposite vertex. Can the spiders catch the fly? (The spiders and the fly see each other at all times.)

Grade 10

33.2.10.1*. A 19-hedron is circumscribed around a sphere of radius 10. Prove that on the surface of the polyhedron there are two points with the distance between them ≥ 21 .

33.2.10.2. Prove that if an integer K is divisible by 10 101 010 101, then there are at least 6 non-zero digits in the decimal expression of K .

33.2.10.3*. See Problem 33.2.9.5, where two spiders are chasing a fly and all have the same top speed.

33.2.10.4. Given an integer $n > 1970$, prove that the sum of the remainders after division of 2^n by 2, 3, 4, \dots , n is greater than $2n$.

33.2.10.5. Merlin has two 100×100 tables; one of them is blank, and on the other table some magic numbers are written. The blank table is nailed to a rock at the entrance to his cave, and the magic one is nailed to a wall inside the cave. You may outline any square (1×1 , 2×2 , \dots , or 100×100) on the blank table, at any place on the table but only along the lines, and for a shilling Merlin will tell you the sum of the numbers of the corresponding square in the magic table. What is the least amount of money one needs to learn the sum of the numbers on the main diagonal of the magic table?

Additional set (Pythagoras' Day)

Grade 7

33.D.7.1. We multiply the number 1234567...1000 (juxtaposed are all natural numbers 1 to 1000) by a number from 1 to 9, and strike out all 1's in the product. We multiply the number obtained by a nonzero one-digit number once again, and strike out the 1's, and so on, many times over. What is the least number one can obtain in this manner?

33.D.7.2. A 13×13 m² hall is divided into squares with sides of 1 m. The Rule requires that rectangular rugs of arbitrary sizes be placed on the floor so that their sides lie on the sides of the squares; in particular, along the side of the hall. Any rug may be partially or even completely covered by other rugs but no single rug may completely cover, or lie under, another rug (even if there are several layers between them). What greatest number of rugs may cover the hall under this Rule?

33.D.7.3. In an ordinary game of dominoes the difference between the numbers on adjacent displayed tiles is equal to 0. Is it possible to arrange all 28 tiles in a closed chain so that the difference throughout the chain would be equal to ± 1 ?

33.D.7.4. Is it possible to divide the numbers 1, 2, 3, \dots , 33 into 11 groups, three numbers in each group, so that in any group one of the numbers is equal to the sum of the other two?

33.D.7.5. Ali Baba tries to enter the cave. At the entrance to the cave there is a drum with four holes in its sides. Inside the drum, next to each hole, there is a switch which has two positions, “up” and “down”. The Rule permits Ali Baba to stick his fingers into any two holes, learn the position of their switches (by touch) and flip them as he pleases (for example not to flip at all). Then the drum is rotated very quickly so that after it stops it is impossible to ascertain which switches were flipped or touched last. Ali Baba may repeat the operation up to 10 times. The door to the cave opens the moment all the switches are in the same position. Prove that Ali Baba can get into the cave.

33.D.7.6. It is known that objects A and B cannot both fit into the picture taken by a camera at point O if $\angle AOB > 179^\circ$. There are 1000 such cameras on a plane. All cameras simultaneously take a picture each. Prove that among these pictures there is one photo that shows ≤ 998 cameras.

Olympiad 34 (1971)

Tour 34.1

Grade 8

34.1.8.1. A town is walled in a 1000-gon (not necessarily convex but with nonintersecting sides). A guard stands at every vertex outside the wall. Prove that there is a guard who can see < 500 other guards. (The guards standing at the endpoints of one side of the 1000-gon can see each other.)

34.1.8.2. A circle intersects convex pentagon $ABCDE$ at points $A_1, A_2, B_1, B_2, \dots, E_1, E_2$; see Fig. 76. Knowing that $AA_1 = AA_2, BB_1 = BB_2, CC_1 = CC_2, DD_1 = DD_2$, prove that $EE_1 = EE_2$.

FIGURE 76. (Probl. 34.1.8.2)

FIGURE 77. (Probl. 34.1.9.2)

34.1.8.3. 25 teams took part in a national soccer tournament. In the end it turned out that no team scored more than four goals in any game. What lowest place could the team from Tbilisi have gotten, if overall it scored more goals, and was scored less goals against, than any other team?

34.1.8.4. A 100×100 square is drawn on a graph paper. There is a red or blue point in every square (of the grid) so that in every column and in every row there are 50 blue and 50 red points. Let us connect every pair of red points in adjacent squares (squares with a common side) with a red segment, and every pair of blue points in adjacent squares with a blue segment. Prove that the number of red segments equals the number of blue segments.

34.1.8.5. Prove that $k(5^{1\ 090\ 701}) - k(2^{1\ 090\ 701}) : 2$, where $k(A)$ is the number of digits in the decimal expression of A .

Grade 9

34.1.9.1. Numbers $a_1, a_2, a_3, \dots, a_{25}$, where $a_1 = a_2 = \dots = a_{13} = 1$, and $a_{14} = a_{15} = \dots = a_{25} = -1$ are written at the vertices of a regular 25-gon. Set $b_1 = a_1 + a_2, b_2 = a_2 + a_3$, etc., $b_{24} = a_{24} + a_{25}, b_{25} = a_{25} + a_1$, and replace a_1, a_2, \dots, a_{25} , with b_1, b_2, \dots, b_{25} , respectively. This operation is then repeated 100 times. Prove that one of the numbers obtained in the operation is greater than 10^{20} .

34.1.9.2. The perimeter of a convex k -gon P ($k > 6$) is equal to 2. Construct a new convex k -gon M with vertices at the midpoints of the sides of the k -gon P and prove that the perimeter of M is greater than 1; see Fig. 77.

34.1.9.3. Consider n straight lines ($n > 2$) on a plane. No two lines are parallel and no three of them meet. It is possible to rotate the plane about some point O through an angle of $\alpha < 180^\circ$ so that each of the lines drawn gets identical with another of lines drawn on the fixed copy of the plane. Find all values of n for which such a system of lines exists.

34.1.9.4. Prove that no integer obtained by permutation of the digits in the decimal expression of 2^k ($k > 3$), is equal to 2^n for $n > k$. (Obviously, n and k are integers here.)

34.1.9.5. Prove that there are infinitely many non-primes among the numbers $[2^k \cdot \sqrt{2}]$, $k = 1, 2, \dots$

Grade 10

34.1.10.1. Consider a closed broken line $A_1A_2 \dots A_nA_1$ in space such that every segment of it intersects a fixed sphere at two points, and all vertices of the line are located outside the sphere. The intersection points divide the broken line into $3n$ segments so that the segments at the vertex A_1 are equal and the same holds true for the vertices A_2, A_3, \dots, A_{n-1} . Prove that the segments at A_n are also equal. (Cf. Problem 34.1.8.2).

34.1.10.2. Peter has a set “Young tiler” of tiles arranged in a rectangular box so that they completely cover the bottom of the box in one layer. Every tile has an area of 3 cm and is either a rectangle or an L-shaped figure, see Fig. 78.

FIGURE 78. (Probl. 34.1.10.2)

Peter says that he lost an L-shaped tile, made a new rectangular tile instead of it, and arranged all tiles in the box in one layer. Can one be certain that Peter is lying?

34.1.10.3. The terms a sequence $x_1, x_2, \dots, x_n, \dots$ satisfy the equation $3x_n - x_{n-1} = n$ for any $n > 1$ and $|x_1| < 1971$. Find x_{1971} to the nearest millionth.

34.1.10.4. All vertices of a convex n -gon and k more points inside it are marked. It turns out that any three of these $n+k$ points are not on the same straight line and are the vertices of an isosceles nondegenerate triangle. What value may k take?

34.1.10.5. There is a pile of 10 million matches. Two players play a game, taking turns. A player may take p^n matches, where p is a prime and $n = 0, 1, 2, 3, \dots$ (for example, the first takes 25 matches, the second takes 8, the first 1, the second 5, the first 49, and so on). The player who takes the last match is the winner. Who wins if both play optimally?

Tour 34.2

Grade 7

34.2.7.1. Is there a number whose square begins with the digits 123456789 and ends with the digits 987654321?

34.2.7.2. Consider square $ABCD$, a point O inside it and perpendiculars AH_1, BH_2, CH_3, DH_4 dropped from points A, B, C, D to segments BO, CO, DO, AO , respectively. Prove that the straight lines on which these perpendiculars lie meet at one point.

34.2.7.3. A colony of n bacteria lived in a beaker. Once, a virus got into the beaker. In the first minute the virus destroyed one bacterium and immediately after that both the virus and the remaining bacteria split in halves. In the second minute the two viruses destroyed two new bacteria, and then the viruses and the remaining bacteria again split in halves, and so on. Will a moment come when no bacteria are left?

34.2.7.4. There is a mesh of 1×1 squares. Its every node is painted one of four given colors so that the nodes of any 1×1 square are differently colored. Prove that there is a straight line of the grid such that the nodes lying on it are painted only two colors.

34.2.7.5*. On a plane stand 7 point-size searchlights. Every searchlight illuminates an angle of 90° . If there is a searchlight in a quadrant illuminated by another searchlight, then the first one casts a shadow, a dark infinite ray. Prove that it is possible to arrange these 7 searchlights so that every one of them will cast a shadow of 7 km long; see Fig. 79.

FIGURE 79. (Probl. 34.2.7.5)

Grade 8

34.2.8.1. Consider a 29-digit number $X = \overline{a_1 a_2 \dots a_{28} a_{29}}$ such that for every k the digit a_k occurs a_{30-k} times in the expression of X . (For example, if $a_{10} = 7$, then the digit a_{20} occurs 7 times.) Find the sum of the digits of X .

34.2.8.2. We cut a cardboard (perhaps, non-convex) 1000-gon along a straight line. This cut brakes it into several new polygons. What is the greatest possible number of triangles among the new polygons?

34.2.8.3. Prove that the sum of the digits of a positive integer K is not more than 8 times the sum of the digits of the number $8K$.

34.2.8.4. Take any number consisting of zeros and fours in its decimal expression. Now, we can either divide it by 2, 3 or 5 if this division is possible without a remainder, or insert 0's or 4's between the digits of this number, or write a 4 at the beginning or at the end, or write a 0 at the end. With the number obtained we can repeat the same operations, and so on. Is it possible to obtain in this way any positive integer?

34.2.8.5. See Problem 34.2.7.2.

Grade 9

34.2.9.1. A convex 1971-gon is such that for every vertex A , every side that does not pass through A subtends equal angles with the angle's vertex in A . Prove that the polygon is a regular one.

34.2.9.2. See Problem 34.2.8.1.

34.2.9.3. Is it possible to divide every side of a square into 100 parts so that it would be impossible to contour with these 400 segments any rectangle other than the initial square?

34.2.9.4. A circle is divided into n equal parts, and the numbers x_1, x_2, \dots, x_n equal to either 1 or -1 are written at the division points so that if one turns the circle through an angle of $k \cdot \frac{360^\circ}{n}$ and multiplies the numbers at points coinciding before and after the rotation, the sum of n products thus obtained is equal to 0 for any $k = 1, \dots, n-1$. Prove that n is a perfect square. (Cf. Problem 34.2.10.1.)

34.2.9.5. Prove that it is possible to write non-zero real numbers x_1, x_2, \dots, x_n at the vertices of a regular n -gon so that for any regular k -gon all of whose vertices are the vertices of the original n -gon, the sum of the numbers at its vertices is equal to 0.

Grade 10

34.2.10.1. See Problem 34.2.9.4 with additional question: what might number n be?

34.2.10.2. Given numbers a_1, \dots, a_n and b_1, \dots, b_n , arrange the numbers a_k in the increasing order and numbers b_k in the decreasing order. We get sets $a_1^* \leq \dots \leq a_n^*$ and $b_1^* \geq \dots \geq b_n^*$. Prove that

$$\max(a_1 + b_1, a_2 + b_2, \dots, a_n + b_n) \geq \max(a_1^* + b_1^*, a_2^* + b_2^*, \dots, a_n^* + b_n^*).$$

34.2.10.3. Banker and Gambler play the following hazardous game. Banker names a 1000-digit number, A_1 . Upon learning this number, Gambler suggests to Banker an arbitrary number B_1 . This ends the first move.

Next, Banker either subtracts the smaller number from the greater, or adds one to the other, as he chooses, and tells the result — a number A_2 — to the Gambler. Then Gambler suggests to Banker the next number, B_2 . Banker repeats the same operation with numbers A_2 and B_2 , and so on.

The game ends when Banker gets one of the following numbers: 1, 10, 100, 1000, \dots . Prove that Gambler can always end the game suggesting not more than 20 of his numbers to Banker.

34.2.10.4. A point O and n straight lines, no two of which are parallel, are given in space. We take the projections of O to all given straight lines. Each of the points obtained is projected to all straight lines again, and so on. Is there a sphere containing inside it all points obtained in such a way?

34.2.10.5*. Prove that the sum of the digits of an integer N is not more than five times the sum of the digits of $5^5 \cdot N$. (Cf. Problem 34.2.8.3).

Olympiad 35 (1972)

Tour 35.1

Grade 7

35.1.7.1. Prove that if positive integers a_1, a_2, \dots, a_{17} satisfy

$$a_1^{a_2} = a_2^{a_3} = a_3^{a_4} = \dots = a_{16}^{a_{17}} = a_{17}^{a_1},$$

then $a_1 = a_2 = \dots = a_{17}$.

35.1.7.2. 1000 delegates from various countries came to a Congress. Every delegate could speak several languages and it was known that any three delegates could have a common conversation without assistance. (A delegate could serve as a translator for a pair of his colleagues.) Prove that it was possible to distribute all delegates in 500 rooms, so that in every room there were 2 delegates and they can understand each other.

35.1.7.3. Every vertex of a regular 13-gon is painted either black or white. Prove that there exist three points of the same color which are the vertices of an isosceles triangle.

35.1.7.4. Let AD and BE be medians in triangle ABC ; let the angles CAD and CBE be equal to 30° . Prove that $AB = BC$. (See Problem 35.1.8.5.)

Grade 8

35.1.8.1. There are asterisks in some of the squares of an $n \times n$ graph paper. It is known that if we strike out any set of rows (but not all of them), a column with exactly one asterisk will remain (if one does not strike out any row there still remains a column with exactly one asterisk). Prove that if one strikes out any number of columns (but not all of them), a row with exactly one asterisk will remain.

35.1.8.2. Given two identical L-shaped figures on a plane. Denote the endpoints of their shorter sides by A and A' and divide their longer sides into n equal parts by points $a_1, \dots, a_{n-1}; a'_1, \dots, a'_{n-1}$. (We number these dividing points beginning at the free endpoints of the longer sides.) Draw the straight lines $Aa_1, Aa_2, \dots, Aa_{n-1}$ and $A'a'_1, A'a'_2, \dots, A'a'_{n-1}$ and denote the intersection point of lines Aa_1 and $A'a'_1$ by X_1 ; of lines Aa_2 and $A'a'_2$ by X_2 , and so on. Prove that points X_1, X_2, \dots, X_{n-2} are vertices of a convex polygon.

35.1.8.3. A pawn got a tip that out of 1000 coins the robber brought him, 0, 1 or 2 are counterfeit. It is known that all counterfeit coins are of the same weight different from the weight of genuine coins. Is it possible to determine (a) whether there are counterfeit coins in this set and (b) whether their weight is greater or less than the weight of genuine coins by weighing groups of coins three times on a balance without using weights? (It is not necessary to determine how many counterfeit coins are there.)

35.1.8.4. Given a set of positive integers with the sum of any seven of them less than 15 and the sum of all the numbers in the set equal to 100, determine the least possible number of elements in this set.

35.1.8.5. Let AD and BE be medians in triangle $\triangle ABC$; let each of the angles $\angle CAD$ and $\angle CBE$ be equal to 30° . Prove that triangle $\triangle ABC$ is an equilateral one.

Grade 9

35.1.9.1. Angle $\angle C$ in triangle ABC is obtuse. Points E and H are marked on side AB and points K and M on sides AC and BC , respectively. It turns out that $AH = AC, EB = BC, AE = AK, BH = BM$. Prove that points E, H, K, M lie on the same circle.

35.1.9.2. There are numbers in all squares of an $n \times n$ chessboard: number a_{km} stands in the intersection of the k -th row with the m -th column. Suppose that for any arrangement of n rooks on this chessboard such that none can be jumped by another, the sum of the numbers covered by the rooks is equal to 1972. Prove that there exist two sets of numbers x_1, x_2, \dots, x_n and y_1, y_2, \dots, y_n that for every k and m satisfy the equation: $a_{km} = x_k + y_m$; cf. Problem 18.1.9.1.

35.1.9.3. The distance between any two trees in a forest is not greater than the difference between their heights. None of the trees is higher than 100 m. Prove that it is possible to fence the forest with a fence 200 m long.

35.1.9.4*. Positive integers m and n are relatively prime and $n < m$. Which number is greater: $\left[1 \cdot \frac{m}{n}\right] + \left[2 \cdot \frac{m}{n}\right] + \dots + \left[n \cdot \frac{m}{n}\right]$ or $\left[1 \cdot \frac{n}{m}\right] + \left[2 \cdot \frac{n}{m}\right] + \dots + \left[m \cdot \frac{n}{m}\right]$?

35.1.9.5. In town X , ten infinite parallel avenues cross perpendicular streets at equal intervals. Two cops moving along the avenues and streets try to find a robber who, according to the Rule, can not shelter in a house and is hiding behind the houses. If the robber turns up on an avenue or street with a cop, he is *found*. The robber's speed is not more than 10 times that of a cop and an informer tipped the cops that the distance between them and the robber at the beginning of the chase was not greater than 100 blocks. Prove that the cops can find the robber.

Grade 10

35.1.10.1. There are n inhabitants in town Variety. Every two of them are either friends or enemies. Every day not more than 1 inhabitant may turn a new leaf: quarrel with all his friends and befriend all his enemies. The Rule of Variety says: if A is a friend of B and B is a friend of C , then A is also a friend of C . Prove that all inhabitants of the town can become friends.

35.1.10.2. Given an infinite sequence $a_1, a_2, \dots, a_n, \dots$, where a_1 is an arbitrary 10-digit number and each subsequent number is obtained from the preceding one by writing any digits but 9 after it, prove that there are no fewer than two non-prime numbers in the sequence.

35.1.10.3. In tetrahedron $ABCD$ all dihedral angles are acute and all opposite edges are equal. Find the sum of the cosines of all dihedral angles of the tetrahedron.

35.1.10.4. Consider a non-selfintersecting non-convex n -gon P and the locus T of points inside P from which one can see all the vertices of P . Prove that if T is nonempty and does not lie on one straight line, then T is a convex k -gon with $k \leq n$.

35.1.10.5. See Problem 35.1.9.5.

Tour 35.2

Grade 7

35.2.7.1. Consider a convex quadrilateral $ABCD$ and point O where its diagonals meet. The perimeters of triangles $\triangle ABO$, $\triangle BCO$, $\triangle CDO$, $\triangle ADO$ are equal. Prove that $ABCD$ is a rhombus.

35.2.7.2. Four straight lines a, b, c, d are drawn on a plane. No two of them are parallel and no three of them meet at one point. It is known that a is parallel to one of the medians of the triangle formed by lines b, c , and d . Prove that b is parallel to a median of the triangle formed by lines a, c , and d .

35.2.7.3. Given twelve consecutive positive integers. Prove that at least one of them is smaller than the sum of its proper divisors.

35.2.7.4*. There are several castles in country Mara and three roads lead from every castle. A knight leaves his castle. Traveling around the country he leaves every new castle via the road that is either to the right or to the left of the one by which he arrived. According to The Rule the knight never takes the same direction (right or left) twice in a row. Prove that some day he will return to his own castle.

35.2.7.5. A straight line intersects sides AB and BC of triangle ABC at points M and K , respectively. Knowing that the area of triangle MBK is equal to the area of quadrilateral $AMKC$, prove that $\frac{MB+BK}{AM+CA+KC} \geq \frac{1}{3}$.

Grade 8

35.2.8.1. See Problem 35.2.7.1.

35.2.8.2. Numbers a, b, c, d, e and f are positive integers such that $\frac{a}{b} > \frac{c}{d} > \frac{e}{f}$ and $af - be = 1$. Prove that $d \geq b + f$.

35.2.8.3. A town of Nikitovka had only two-way traffic. Repairs of all its streets took two years. During the first year some of the streets were turned into one-way streets. The next year the two-way traffic was reestablished on these roads whereas all other roads became one-way roads. The repairs were made under strict adherence to the following Rule: *one should be able to drive from any point of the town to any other at all times during the repairs.* Prove that it is possible to introduce a one-way traffic throughout Nikitovka so that one could still drive from any point of the town to any other point.

35.2.8.4. Let $I(x)$ be the number of irreducible fractions $\frac{a}{b}$, where both a and b are positive integers such that $a \leq x$ and $b \leq x$. For example, $I\left(\frac{5}{2}\right) = 3$ and the corresponding fractions are $\frac{1}{2}$; 1; 2. Find the sum:

$$I(100) + I\left(\frac{100}{2}\right) + I\left(\frac{100}{3}\right) + \dots + I\left(\frac{100}{99}\right) + I\left(\frac{100}{100}\right).$$

35.2.8.5. See Problem 35.2.9.5 for 300 straight lines and 100 triangles.

Grade 9

35.2.9.1. All sides of a pentagon are of the same length and each of its angles is less than 120° . Prove that all its angles are obtuse.

35.2.9.2. See problem 35.2.8.2.

35.2.9.3*. The streets of town M form a regular square net of 20×20 blocks. There are subway stations at some corners. It is known that one can get to a subway station from any point in the town passing not more than two blocks along the streets. What is the least number of subway stations in the town?

35.2.9.4*. Are there any rational numbers a, b, c, d satisfying for a positive integer n the equation

$$(a + b\sqrt{2})^{2n} + (c + d\sqrt{2})^{2n} = 5 + 4\sqrt{2} ?$$

35.2.9.5*. 3000 straight lines are drawn on a plane, no two of them are parallel, and no three of them meet at the same point. These lines divide the plane into several parts. Prove that among these parts there are at least a) 1000 triangles; b*) 2000 triangles.

Grade 10

35.2.10.1*. Consider plane P and triangle ABC , not on this plane, see Fig. 80. Triangle $A_1B_1C_1$ is a perpendicular projection of triangle ABC to P . Prove that triangle $A_1B_1C_1$ can be completely covered by a triangle equal to triangle ABC .

FIGURE 80. (Probl. 35.2.10.1)

35.2.10.2. Given two sets of numbers x_1, x_2, \dots, x_n and y_1, y_2, \dots, y_n such that

$$x_1 > x_2 > \dots > x_n > 0, \quad y_1 > y_2 > \dots > y_n > 0;$$

and

$$x_1 > y_1, \quad x_1 + x_2 > y_1 + y_2, \quad \dots, \quad x_1 + x_2 + \dots + x_n > y_1 + y_2 + \dots + y_n,$$

prove that for any positive integer k we have

$$x_1^k + x_2^k + \dots + x_n^k > y_1^k + y_2^k + \dots + y_n^k.$$

35.2.10.3. Numbers 1, 2, 3, \dots , 400 are written on 400 cards, one on each card. Players A and B play the following game. Player A selects any 200 cards (the first set) and gives the rest (the second set) to B . Then B takes 100 cards from each set and gives the rest to A . Thus, both players once again have 200 cards. The end of the first move.

Then A again takes 100 cards from each set and gives the rest to B , and so on. After B has moved for the 200-th time, both players count the sum of the numbers on their cards, $S(A)$ and $S(B)$; and A pays B the difference $S(B) - S(A)$. What greatest amount of money can B get if both play optimally?

35.2.10.4. Arrange all rational numbers between zero and one whose denominators do not exceed n in increasing order. Let irreducible fractions $\frac{a}{b}$ and $\frac{c}{d}$ be two consecutive such numbers. Prove that $|bc - ad| = 1$ whatever n .

35.2.10.5*. There is a positive integer in every square of an 8×8 chessboard. The Rule allows one to take any square of size 3×3 or 4×4 and increase all numbers in it by 1 to ensure that the numbers in all squares are divisible by 10. Is this always possible after several such operations?

Olympiad 36 (1973)

Tour 36.1

Grade 8

36.1.8.1. There are several countries on a square island. Is it possible to divide these countries into smaller ones without creating new intersection points of their borders, and so that the map of the island could be painted two colors?

36.1.8.2. Can a number whose decimal expression consists of 600 copies of figure 6 and several zeros be the square of a positive integer?

36.1.8.3. Consider five points in a plane, no three of which lie on the same straight line and no four of which are on the same circle. Prove that two of these points may be selected so that they lie on both sides of the circle passing through three other points.

36.1.8.4. Prove that the equation $\frac{1}{x} + \frac{1}{y} = \frac{1}{p}$, where x, y are positive integers, has exactly 3 solutions if p is a prime and the number of solutions is greater than three if $p > 1$ is not a prime. We consider solutions (a, b) and (b, a) for $a \neq b$ as distinct.

36.1.8.5. On a plane, in three vertices of a square sit three grasshoppers. At some moment the grasshoppers begin playing a game of leap-frog according to the following Rule: they leap over each other so that if grasshopper A leaps over grasshopper B , then after the jump it is at the same distance from B as before and on the same line. Is it possible for any of the grasshoppers to reach the fourth vertex of the square after a few jumps?

Grade 9

36.1.9.1. The area of a quadrilateral with vertices on the sides of a parallelogram is equal to half the area of the parallelogram. Prove that at least one of the quadrilateral's diagonals is parallel to a side of the parallelogram.

36.1.9.2. A square is divided into convex polygons. Prove that it is possible to divide them into smaller convex polygons so that each of these new ones has an odd number of adjacent polygons (with a common side).

36.1.9.3. The value of a polynomial $P(x)$ with integer coefficients is equal to 2 at three integer points. Prove that there exists no integer point at which the polynomial is equal to 3.

36.1.9.4. In the city of X one can get to any subway station from any other. Prove that it is possible to close one of the stations for repairs and not let trains pass through it but still enable people to get to any of the remaining stations from any other.

36.1.9.5. The faces of a cube are numbered 1, 2, \dots , 6 so that the sum of the numbers on every pair of opposite faces is equal to 7. There is a 50×50 chessboard whose squares are equal to the faces of the cube. The cube rolls from the lower left corner of the chessboard to its upper right corner. The Rule allows it to move only to the right or upwards (not to the left or downwards). The cube prints the numbers painted on its faces in every square of the chessboard that a face touches as the cube rolls. What is the greatest possible sum of the numbers printed and what is the least possible one? (The figure 6 printed upside down still counts as 6, not 9!)

Grade 10

36.1.10.1. We factor a positive integer k into its prime factors: $k = p_1 \cdot p_2 \cdot \dots \cdot p_{n-1} \cdot p_n$ and set $f(k) = p_1 + p_2 + \dots + p_{n-1} + p_n + 1$. We perform the same operation f with $f(k)$, and so on. Prove that the resulting sequence of numbers $k, f(k), f(f(k)), \dots$ eventually becomes a periodic one.

36.1.10.2. See Problem 36.1.9.1.

36.1.10.3. At some integer points a polynomial $P(x)$ with integer coefficients takes values 1, 2 and 3. Prove that there exists not more than one integer at which the polynomial is equal to 5.

36.1.10.4. Prove that every convex polyhedron has two faces with the same number of sides.

36.1.10.5. A control panel of N switches and a board with N bulbs are on the sides of a “black box”. By switching consecutively all possible combinations of the switches we consecutively light all possible combinations of the bulbs. The state of the panel of lights directly depends on the state of the switches on the control panel. It is known that when one switch is flipped, exactly one bulb lights up or goes off. Prove that the state of each bulb depends on exactly one switch (for every bulb its own switch).

Tour 36.2

Grade 7

36.2.7.1. A four-digit number is subtracted from a number composed of the same digits in reverse order. Can the difference be equal to 1008?

36.2.7.2. Consider an acute triangle ABC and discs centered at the vertices of the triangle with their radii equal to heights dropped from respective vertices. Prove that every point of the triangle is covered by at least one disc.

36.2.7.3. A 100×100 piece of graph paper is painted 100 different colors. Every unit square of the grid is either painted one of the colors or not painted at all. A coloring will be called *regular* if no column and no row has two squares of the same color. Is it possible to paint this piece of paper regularly so that all squares are painted if initially there are a) $100^2 - 1$; or b) $100^2 - 2$; or c) 100 regularly painted squares?

36.2.7.4. See Problem 36.2.8.3 a) below.

Grade 8

36.2.8.1. There is an ink blot on a piece of paper. For every point of the blot consider its minimal and the maximal distance to the boundary of the blot. The greatest of all minimal distances and the least of all maximal distances are selected and compared. What is the shape of the blot if these two numbers are equal? (See Fig. 81)

FIGURE 81. (Probl. 36.2.8.1)

36.2.8.2. See Problem 36.2.7.3 replacing 100 with an arbitrary n .

36.2.8.3. At the center of a square stands a cop and at one of the square’s vertices stands a robber. The Rule allows the cop to run anywhere in the square and even digress outside its limits, while the robber can only move along the square’s sides. For each of the following ratios of the cop’s top speed to that of the robber a) $1/2$; b) 0.49 ; c) 0.34 ; d) $1/3$, prove that the cop can run so as to be on the same side as the robber at some moment.

36.2.8.4. Prove that it is possible to place an equilateral triangle into a convex equilateral (but not necessarily regular) pentagon, with sides equal to the sides of the triangle, so that they have one side in common and the entire triangle is inside the pentagon.

Grade 9

36.2.9.1. The decimal expression of a 100-digit number consists of 1's and 2's. The Rule allows one to select arbitrarily 10 consecutive digits of which the first five may change places with the second five. Two numbers will be called *similar* if one can be obtained from the other one in several such operations. What greatest number of such 100-digit numbers can be selected so no two of them are similar?

36.2.9.2. A closed non-selfintersecting broken line is drawn on an infinite chessboard along the sides of its squares. There are K black squares inside the broken line. What is the greatest area of the figure bounded by the broken line?

36.2.9.3. See Problem 36.2.10.1 for $m = 5$.

36.2.9.4*. Two 1's are situated at the endpoints of a line segment. The first move is to insert their sum — the number 2 — between them. Next move is to insert between every two adjacent numbers their sum, and so on, 1 000 000 times; see Fig. 82. How many times will the number 1973 be written during this process?

FIGURE 82. (Probl. 36.2.9.4)

36.2.9.5*. See Problem 36.2.8.3. Let the robber's top speed be 2.9 times that of the cop. Is it possible for the cop to arrive on the same side with the robber?

Grade 10

36.2.10.1. Let m and n be positive integers ≥ 2 . Prove that there is a positive integer k such that

$$\left(\frac{n + \sqrt{n^2 - 4}}{2} \right)^m = \frac{k + \sqrt{k^2 - 4}}{2}.$$

36.2.10.2. Prove that the angles between every two bisectors of planar angles of a trihedral angle are either all acute, or all obtuse, or all right ones.

36.2.10.3. 12 painters live in a commune of 12 red and white houses along a ring-shaped road. Every month one of the painters leaves his or her house with red and white paints and goes clockwise along the road. When (s)he sees a red house (s)he paints it white and goes further, and when (s)he sees a white house (s)he paints it red and then goes home to wash his or her brush. Every painter only works once a year. Prove that if at the beginning of the year there is at least one red house then at the end of a year every house will be painted its initial color.

36.2.10.4. See Problem 36.2.9.1.

36.2.10.5. A lion runs over a circular circus ring of radius 10 m. Moving along a broken line he covers 30 km. Prove that the sum of the angles of all of the lion's turns is not less than 2998 radians.

Olympiad 37 (1974)

Tour 37.1

Grade 9

37.1.9.1. Prove that the number 100...001 with $2^{1974} + 2^{1000} - 1$ zeros is not a prime.

37.1.9.2. Prove that it is impossible to place two triangles, each of area greater than 1, into a disc of radius 1 so that they would not overlap.

37.1.9.3. Two identical gears have 32 teeth each. One of the gears was placed atop the other one so that their teeth aligned. Then 6 pairs of corresponding teeth were sawed off from both gears. Prove that it is possible to rotate one gear relative the other one so that in the places where teeth of one gear are missing there will be teeth of the other gear. (Cf. Problem 37.1.10.3.)

37.1.9.4. Prove that if it is possible to construct a triangle from segments of lengths a , b and c , it is also possible to construct a triangle from segments of lengths $\frac{1}{a+c}$, $\frac{1}{b+c}$, $\frac{1}{a+b}$.

37.1.9.5. A convex polygon has the following property: if all straight lines on which its sides lie are moved outwards by a distance of 1, then the straight lines in their new positions form a new polygon similar to the original one, with the proportional parallel sides. Prove that it is possible to inscribe a circle into the original polygon.

Grade 10

37.1.10.1. See Problem 37.1.9.4.

37.1.10.2. Prove that for any 13-gon there exists a straight line which contains exactly one of its sides but for any $n > 13$ there exists such an n -gon for which this does not hold.

37.1.10.3. Two identical gears have 92 teeth each. One of the gears was placed atop the other one so that their teeth aligned. Then 10 pairs of corresponding teeth were sawed off from both gears. Prove that it is possible to rotate one gear relative the other one so that in the places where teeth of one gear are missing there will be teeth of the other gear. (Cf. Problem 37.1.9.3.)

37.1.10.4. Suppose we mark all vertices and centers of the faces of a cube and draw all diagonals of its faces. Is it possible, moving along the diagonals, to pass every marked point only once?

37.1.10.5. See Problem 37.1.9.5.

Tour 37.2

Grade 7

37.2.7.1. Point M inside a regular hexagon with side 1 is connected with all vertices of the hexagon thus dividing the hexagon into triangles. Prove that among the triangles there are two whose sides are not shorter than 1.

37.2.7.2. On a straight line 100 points are fixed. Let us mark the midpoints of all segments with both endpoints among the fixed points. What is the minimal number of marked points? (Cf. Problem 37.2.8.2.)

37.2.7.3. How many sides can a convex polygon have if its diagonals are of equal length?

37.2.7.4. A few marbles are distributed into three piles. A boy who has an access to an unlimited stock of marbles may take one marble from every pile or add to one of the piles as many marbles from his stock as there are already in the pile. Prove that in a few such operations the boy can make it so that there are no marbles left in every pile.

Grade 8

37.2.8.1. See Problem 37.2.7.3.

37.2.8.2. On a straight line n points are fixed. Let us mark the midpoints of all segments with both endpoints among the fixed points. What is the minimal number of marked points? (Cf. Problem 37.2.7.2.)

37.2.8.3. Positive integers fill in a rectangular table of 8 rows and 5 columns. In one move we may double all numbers in one row or subtract 1 from every number in one column. Prove that it is possible to make all the numbers in the table equal to 0 in finitely many moves.

37.2.8.4. Prove that a convex pentagon with all angles obtuse has two diagonals such that discs constructed on them as on diameters completely cover the pentagon.

37.2.8.5. The sum of 100 positive integers, each not greater than 100, is equal to 200. Prove that from these integers one can select several so that their sum is equal to 100.

Grade 9

37.2.9.1. Is there a sequence of positive integers such that one can uniquely express any positive integer $1, 2, 3, \dots$, as the difference of two numbers of the sequence?

37.2.9.2. Prove that in an arbitrary $2n$ -gon there exists a diagonal not parallel to any of its sides.

37.2.9.3*. There are several weights of (positive) integer masses. It is known that they can be divided into K groups of equal mass. Prove that in not less than K ways one can take away a weight so that it is impossible to divide the remaining weights into K groups of equal mass.

37.2.9.4. Given triangle ABC with $AB > BC$ and its bisectors AK and CM , prove that $AM > MK > KC$. (See Solution to Problem 28.1.9.3.)

37.2.9.5. An $a \times b$ piece of paper is cut into rectangular strips, each one with a side of 1 cm. The cuts are parallel to the edges of the paper. Prove that at least one of the numbers a or b is an integer.

Grade 10

37.2.10.1. See Problem 37.2.9.1.

37.2.10.2. Prove that the decimal expressions of the numbers $2^n + 1974^n$ and 1974^n have the same number of digits.

37.2.10.3. A spherical planet is surrounded by 37 point-size asteroids. An asteroid on the horizon is invisible. Prove that at any moment of time there is a point on the surface of the planet from which an astronomer cannot see more than 17 asteroids.

37.2.10.4. Scientists, some of whom are acquainted, come to a congress. It turns out that no two scientists with the same number of acquaintances have any acquaintances in common. Prove that there is a scientist who has exactly one acquaintance among the participants of the congress.

37.2.10.5. See Problem 37.2.9.5.

Olympiad 38 (1975)

Tour 38.1

Grade 10

38.1.10.1. Solve in real numbers

$$x^2 + y^2 + z^2 + t^2 = x(y + z + t).$$

38.1.10.2. The distance between the center of a disc of radius 1 cm and a point A is 50 cm. We can symmetrically reflect point A through any straight line intersecting the disc; any point obtained may also be reflected symmetrically through any straight line intersecting the disc, and so on. Prove that a) it is possible to herd point A inside the disc in 25 reflections; b) it is impossible to do so in 24 reflections.

38.1.10.3. Positive integers a, b, c are such that the numbers $p = b^c + a$, $q = a^b + c$, and $r = c^a + b$ are primes. Prove that two of the numbers p, q, r are equal.

38.1.10.4. The centers of the squares of an 8×8 chessboard — 64 points — are marked. Is it possible to separate every marked point from the rest by drawing 13 straight lines that do not intersect these points?

38.1.10.5*. Is it possible to arrange 4 lead balls and a point source of light in space so that every ray of light from the source would end in at least one of the balls?

Tour 38.2

Grade 7

38.2.7.1. See Problem 38.2.8.1 a) where $n = 100$.

38.2.7.2. A convex heptagon is inscribed in a circle. It is known that three of the heptagon's angles are equal to 120° . Prove that two of the heptagon's sides are of the same length.

38.2.7.3. Kolya and Vitya play the following game. There is a pile of 31 stones on the table. The boys take turns making moves and Kolya begins. In one turn a player divides every pile which has more than one stone into two lesser ones. The player who after his turn leaves all piles with only one stone in each wins. Can Kolya win no matter how Vitya plays?

38.2.7.4. In the sequence $19752\dots$ every digit beginning with the fifth one is equal to the last digit of the sum of the preceding four digits. Is it possible to find in the sequence a) strings of consecutive digits 1234? 3269? b) a second string 1975?

Grade 8

38.2.8.1. Which of the two numbers is greater:

a) $2^{\overset{2}{\vdots}} (n \text{ many } 2\text{'s})$ or $3^{\overset{3}{\vdots}} (n - 1 \text{ many } 3\text{'s})?$

b) $3^{\overset{3}{\vdots}} (n \text{ many } 3\text{'s})$ or $4^{\overset{4}{\vdots}} (n - 1 \text{ many } 4\text{'s})?$

38.2.8.2. See Problem 38.2.7.2.

38.2.8.3. See Problem 38.2.7.4 with the addition: c) the set 8197?

38.2.8.4. There are two countries: Ourland and the Behind the Looking Glass, or just the Behindland. Every town in Ourland has its “double” in the Behindland and vice versa. If some two towns A and B are connected by a railroad in Ourland, then their doubles A' and B' are not connected in the Behindland, but the doubles of two unconnected towns of Ourland are connected by a railroad in the Behindland. A girl Alice from Ourland cannot reach town B from town A changing trains fewer than two times. Prove that her double, Ecila, in the Behindland can get from one town to any other changing trains not more than twice.

38.2.8.5. In a soccer tournament n teams take part. Every team plays with the other one only once. What can the greatest difference between the final scores of the team with neighboring final positions be?

Grade 9

38.2.9.1. See Problem 38.2.8.1.

38.2.9.2. See Problem 38.2.7.2.

38.2.9.3. See Problem 38.2.8.5.

38.2.9.4. In the land Mantissa towns are connected by roads. The length of any road is less than 500 km, and it is possible to get from any town to any other one driving less than 500 km. When one of the roads was closed for repairs it turned out that it was still possible to get from any town to any other one. Prove that in this case one can find a road between any two towns not longer than 1500 km.

38.2.9.5*. Is it possible to cut a convex polygon into a finite number of non-convex quadrilaterals? (See Fig. 83.)

FIGURE 83. (Probl. 38.2.9.5)

Grade 10

38.2.10.1. See Problem 38.2.8.1.

38.2.10.2. See Problem 38.2.7.3 and replace 31 with 100.

38.2.10.3. See Problem 38.2.9.4.

38.2.10.4*. Several ($n > 0$) distinct spotlights illuminate a circus ring in the form of a disc. Every spotlight illuminates some convex lamina on the ring. It is known that if any of spotlights is turned off the ring is still fully illuminated, and if 2 arbitrary spotlights are turned off the ring is not fully illuminated. For what n this is possible?

38.2.10.5. See Problem 38.2.9.5.

Olympiad 39 (1976)

Tour 39.1

Grade 10

39.1.10.1. Find all positive solutions of the system of equations:

$$\begin{cases} x_1 + x_2 = x_3^2, \\ x_2 + x_3 = x_4^2, \\ x_3 + x_4 = x_5^2, \\ x_4 + x_5 = x_1^2, \\ x_5 + x_1 = x_2^2. \end{cases}$$

39.1.10.2. We drew median AM , bisector BK and height CH in an acute triangle $\triangle ABC$. Let $\triangle M'H'K'$ be the triangle formed by the intersection points of the three segments drawn. Can it be so that $S_{\triangle M'H'K'} > 0.499 \cdot S_{\triangle ABC}$? (See Fig. 84.)

FIGURE 84. (Probl. 39.1.10.2)

39.1.10.3. In the decimal expression of $1^1 + 2^2 + 3^3 + \dots + 999^{999} + 1000^{1000}$, what are its a) first three digits from the left? b) first four digits?

39.1.10.4. An astronomical searchlight can illuminate an octant. The searchlight stands at the center of a cube. Is it possible to turn the searchlight so that it will not illuminate any of the cube's vertices?

39.1.10.5. Domino tiles 1×2 are placed on an infinite graph paper composed of unit squares. The tiles cover all squares. Can it be so that any straight line of the graph of the paper intersects only a finite number of tiles?

Tour 39.2

Grade 7

39.2.7.1. There are four balls, identical in appearance, of mass 101 g, 102 g, 103 g, and 104 g. The Rule allows you to use a balance with two pans and an arrow (indicating weight on a continuous scale). The balance can measure any weight. Find the mass of every ball in only two weighings.

39.2.7.2. Can a convex non-regular pentagon have exactly 4 sides of equal length and exactly 4 diagonals of equal length?

39.2.7.3. Is there a positive integer n such that the sum of the digits of the number n^2 is equal to 100?

39.2.7.4. Is it possible to fix finitely many points on a plane so that every fixed point has exactly 3 nearest neighboring points? (Cf. Problem 32.2.10.4.)

39.2.7.5. There are 200 distinct numbers arranged in a 10×20 table. The two greatest numbers of every row are marked red and the two greatest numbers of every column are marked blue. Prove that at least three (skoljko točno?) numbers are marked both red and blue. (Cf. Problem 32.2.10.3.)

Grade 8

39.2.8.1. See Problem 39.2.7.3.

39.2.8.2. The length of the side of square $ABCD$ is an integer. Line segments parallel to the square's sides divide it into smaller squares; the lengths of the sides of the smaller squares are also integers. Prove that the sum of the lengths of all segments is divisible by 4.

39.2.8.3. See Problem 39.2.9.2.

39.2.8.4. See Problem 39.2.7.5.

39.2.8.5. See Problem 39.2.10.4.

Grade 9

39.2.9.1. Is there an integer n such that $n!$ terminates with the digits 1976000...000 (the number of zeroes is not specified)? (I.e., if you find such an n for any number of zeros, you have answered in affirmative, otherwise you have to prove that whatever number of zeros, there is no such n .)

39.2.9.2. On the spherical Sun finitely many circular spots are discovered. Each spot covers less than half of the Sun's surface. The spots are considered to be *closed* (i.e., a spot's boundary belongs to it) and they neither intersect nor touch one another. Prove that on the Sun there are two diametrically opposite points not covered by the spots. (See Fig. 85.)

FIGURE 85. (Probl. 39.2.9.2)

39.2.9.3*. Prove that there exists a positive integer n greater than 1000 such that the sum of the digits in the decimal expression of 2^n is greater than same of 2^{n+1} .

39.2.9.4. There are no zeros in the decimal expression of a given number N . If two identical digits or two identical two-digit numbers neighbor in the decimal expression of N , we may strike them out. Besides, we are allowed to insert two identical neighboring digits or two identical neighboring two-digit numbers into any place in the decimal expression of N . Prove that with such operations we can obtain from N a number less than 10^9 .

39.2.9.5*. On a table, there is a vast piece of graph paper (the side of each square of the grid being 1 cm). There is also an unlimited number of 5-kopek coins of radius 1.3 cm. Prove that it is possible to put the coins on the paper so that they cover all nodes of the graph but do not overlap.

Grade 10

39.2.10.1. Is there a positive integer A such that \overline{AA} is a perfect square?

39.2.10.2. Is there a convex 1976-hedron such that for an arbitrary arrangement of arrows, one on each edge, the sum of the vectors the arrows represent is not equal to $\vec{0}$?

39.2.10.3. There are 200 different numbers arranged in a 10×20 table. The three greatest numbers of each row are marked red, and the three greatest numbers of each column are marked blue. Prove that at least 9 numbers are marked both red and blue. (Cf. Problem 39.2.7.5.)

39.2.10.4. On a plane, there are fixed several (finitely many) points. For every fixed point A consider the shortest distance r from A to any other fixed point; a fixed point at distance r from A is called a *neighbor* of A . Prove that there is a fixed point with not more than three neighbors. (Cf. Problem 39.2.7.4.)

39.2.10.5. Every point in space is painted one of five given colors, and there are fixed 5 points painted different colors. Prove that there exists a straight line all whose points are painted not less than three colors, and a plane all whose points are painted not less than four colors.

Olympiad 40 (1977)

Tour 40.1

Grade 10

40.1.10.1. A sequence is determined by recurrence: $x_1 = 2$, $x_{n+1} = \left\lceil \frac{3}{2}x_n \right\rceil$ for $n > 1$. Prove that the sequence has an infinite set of a) odd numbers; b) even numbers.

40.1.10.2. On a table, n cardboard squares and n plastic squares are arranged. No two cardboard squares have a common point (boundary points included). The same holds for the plastic squares. It turns out that the set of vertices of the cardboard squares coincides with the set of vertices of the plastic squares. Must then every cardboard square coincide with some plastic square?

40.1.10.3*. a) Twelve thin solid wires of length 1 each are joined to form the frame of a unit cube. Is it possible to make in a plane a hole of area ≤ 0.01 , not cutting the plane into several parts, so that the whole frame can be pulled through the hole?

b) The same question for the frame of a tetrahedron with edge of length 1.

40.1.10.4. On the real line every point with integer coordinate is painted either red or blue. Prove that either red or blue the following property: for every positive integer K there is an infinite number of points of this color whose coordinates are divisible by K .

Tour 40.2

Grade 7

40.2.7.1. In every vertex of a convex n -gon lies a hunter with a laser gun. All hunters simultaneously fire at a rabbit sitting in a point O inside this n -gon. At the moment of the shot the rabbit lies down and all hunters get killed¹. Prove that, apart from O , there is no other point with the same property.

40.2.7.2. A $3 \times 3 \times 3$ cube is made of 14 white and 13 black smaller cubes with edge 1. A *stack* is a collection of three smaller cubes standing in a row in one direction: width, length or height. Could there be an odd number of (a) white cubes or (b) black cubes in every stack?

40.2.7.3. Prove that there are more than 1000 three-tuples of positive integers (a, b, c) satisfying $a^{15} + b^{15} = c^{16}$.

40.2.7.4. 1977 nails stick out of a board. Two players make moves taking turns. In one move a player connects two nails with a wire. Two nails previously connected may not be connected again. If a move results in a closed chain, the player who made the move wins. Who wins if both play optimally — the first player or the second one?

40.2.7.5. Find the minimal n such that any convex 100-gon can be obtained as the intersection of n triangles. Prove that for a smaller n not every convex 100-gon can be obtained in this way.

Grade 8

40.2.8.1. See Problem 40.2.7.1.

40.2.8.2. See Problem 40.2.7.2.

40.2.8.3. See Problem 40.2.7.3.

40.2.8.4. See Problem 40.2.9.3 a).

40.2.8.5. See Problem 40.2.7.5.

Grade 9

40.2.9.1. In space there are n segments no three of which are parallel to one plane. For any two of them a straight line connecting their midpoints is perpendicular to both of them. For what greatest n is this possible?

40.2.9.2. a) Are there 6 different positive integers such that $(a + b) \mid (a - b)$ for any two of them, a and b ?

b) The same question for 1000 numbers.

40.2.9.3. a) At the end of a volleyball tournament it turned out that for any two teams there was a third one which had beaten both of them. Prove that the number of teams in the tournament was ≥ 7 .

b) In another volleyball tournament for any three teams there was a team which had beaten all three. Prove that the number of teams in this tournament was ≥ 15 .

40.2.9.4. The vertices of a convex polyhedron in space are all situated at integral points (i.e., all three coordinates of every vertex are integers). There are no other integral points either inside the polyhedron or on its faces and edges. Prove that the polyhedron has not more than 8 vertices.

40.2.9.5*. Consider a polynomial $P(x)$ with integer coefficients such that $P(n) > n$ for any positive integer n and such that for every positive integer N the sequence

$$x_1 = 1, \quad x_2 = P(x_1), \quad \dots, \quad x_n = P(x_{n-1}), \dots$$

has a term divisible by N . Prove that $P(x) = x + 1$.

¹Which serves them right: don't get involved into such a problem.

Grade 10

40.2.10.1. Is it possible to place an infinite set of identical discs on a plane so that any straight line on this plane intersects not more than two discs?

40.2.10.2. See Problem 40.2.9.2 for 15 numbers.

40.2.10.3. See Problem 40.2.9.3 b).

40.2.10.4. Consider the recurrence: $x_1 = 2$, $x_{n+1} = \left\lceil \frac{3}{2}x_n \right\rceil$ for $n > 1$. Prove that the sequence $\{y_n = (-1)^{x_n}\}_{n \in \mathbb{N}}$ is non-periodic.

40.2.10.5. See Problem 40.2.9.5.

Olympiad 41 (1978)

Grade 7

41.7.1. Solve in positive integers $3 \cdot 2^x + 1 = y^2$.

41.7.2*. On a plane lies a plastic triangle. If it is rolled over and over its sides and at some moment intersects its initial position, then we know that it simply coincides with its initial position. For what triangles this is true? Indicate all types of such triangles.

41.7.3. Prove that it is possible to arrange dominoes of size 1×2 in two layers on an $n \times 2m$ rectangle ($m, n \in \mathbb{N}$) so that each layer fully covers the rectangle and so that no two dominoes of different layers coincide.

41.7.4. See Problem 41.10.2 a).

Grade 8

41.8.1. See Problem 41.9.1.

41.8.2. See Problem 41.7.2.

41.8.3. See Problem 41.7.3.

41.8.4. See Problem 41.10.2 a).

41.8.5. A 1000-digit natural number A has the following remarkable property. Any 10 of its consecutive digits form a number divisible by 2^{10} . Prove that A is divisible by 2^{1000} .

Grade 9

41.9.1. Several points inside an n -gon are situated in such a way that inside any triangle formed by three vertices of the n -gon there lies at least one of the points. What is the least possible number of these points?

41.9.2. Is there a finite number of vectors $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$ on a plane such that for any pair of distinct vectors of this set there is another pair of vectors of the set whose sum is equal to that of the first pair?

41.9.3. See Problem 41.10.2 below.

41.9.4. In plane, consider several (finitely many) straight lines and points. Prove that there exists a point A on the plane, which does not coincide with any of the given points, and with distance to any given point greater than the distance to any of the given straight line.

41.9.5. There are 100 gossips in a town. Every gossip has 3 friends, also gossipy. A gossip learns some interesting news on the first of January and tells the news to his or her three friends. On the second of January the friends tell the news to every one of their friends, and so on. Is it possible that by the 5-th of March not all gossips have learned the news, but that all of them will have learned it by the 19-th of March?

Grade 10

41.10.1. A white sphere has 12% of its area painted red. Prove that it is possible to inscribe a parallelepiped into the sphere so that all its vertices are white.

41.10.2. A square town has 6 streets: 4 streets are the sides of the square and two are its medians. A cop is chasing a robber in this town. If the cop and the robber arrive at the same street simultaneously, then the robber gives in. Prove that the cop can catch the robber if the cop's top speed is a) 3 times that of the robber; b*) 2.1 times that.

41.10.3. See Problem 41.9.4.

41.10.4*. Prove that there exists a) a positive integer, b) an infinite set of positive integers n such that several consecutive last digits of 2^n in its decimal expression form the number n .

41.10.5*. Given 8 real numbers: a, b, c, d, e, f, g, h , prove that at least one of the six numbers $ac + bd, ae + bf, ag + bh, ce + df, cg + dh, eg + fh$ is non-negative.

Olympiad 42 (1979)

Grade 7

42.7.1. On a plane point O is marked. Is it possible to place on the plane a) 5, b) 4 discs that do not cover O so that any ray originating in O intersects at least two discs?

42.7.2. There are several weights with total mass of 1 kg. The weights are numbered 1, 2, 3, Prove that there is n such that the mass of the n -th weight is greater than 2^{-n} kg.

42.7.3. A square is cut into rectangles. Prove that the sum of areas of the discs circumscribed around the rectangles is not less than the area of the disc circumscribed around the square. (See Fig. 86.)

FIGURE 86. (Probl. 42.7.3)

42.7.4. Kolya and Vitya play the following game on an infinite graph paper. Kolya begins and taking turns they mark nodes of the paper, one node each per move. Both must mark so that after a move all points marked would be the vertices of a convex polygon (beginning with Kolya's second move). The player who cannot make such a move loses. Who wins if both play optimally?

Grade 8

42.8.1. A point O is marked on a plane. Is it possible to place on the plane a) 7 discs, b) 6 discs, that do not cover point O , so that any ray beginning from O intersects at least three discs? (Cf. Problem 42.7.1).

42.8.2. See Problem 42.7.2.

42.8.3. A quadrilateral $ABCD$ is inscribed in a circle with center O . Diagonals AC and BD are perpendicular. Prove that the length of perpendicular OH dropped from the center of the circle to side AD is equal to half the length of side BC . (See Fig. 87.)

FIGURE 87. (Probl. 42.8.3)

42.8.4. See Problem 42.7.3.

42.8.5. k scientists — chemists and alchemists — take part in a conference on chemistry. There are more chemists than alchemists among the scientists. It is known that chemists always tell the truth, no matter what they are asked, and that alchemists sometimes tell the truth and sometimes do not (lie).

A mathematician wants to know about every scientist whether the person in question is a chemist or alchemist. The Rule allows the mathematician ask any scientist the question: "What is such and such: chemist or alchemist?" (referring to any scientist, including the one questioned). Prove that the mathematician can learn what (s)he wants to know in a) $4k$ questions; b) $2k - 2$ questions.

Grade 9

42.9.1. Given a collection of stones. The mass of each stone is ≤ 2 kg and their total mass is equal to 100 kg. We selected a set of stones whose total mass differs from 10 kg by the least possible for this set number d . What is the greatest value of d for every admissible collection of stones?

42.9.2*. Is it possible to represent the whole space as the union of an infinite number of pairwise skew lines?

42.9.3*. a) Does there exist a sequence of positive integers a_1, a_2, a_3, \dots such that none of its elements is equal to the sum of some other ones, and $a_n \leq n^{10}$ for every n ?

b) The same question with $a_n \leq n\sqrt{n}$ for every n .

42.9.4. See Problem 42.8.3.

42.9.5*. See Problem 42.8.5 with a new heading: c) $2k - 3$ questions.

Grade 10

42.10.1. See Problem 42.9.1.

42.10.2. On a segment of length 1 several intervals are marked. It is known that the distance between any two points from the same or different marked intervals is not equal to 0.1. Prove that the sum of lengths of the marked intervals is not greater than 0.5.

42.10.3. A function $y = f(x)$ is defined and is twice differentiable on segment $[0, 1]$. Moreover, $f(0) = f(1) = 0$ and $|f''(x)| \leq 1$ on the whole segment. What is greatest value of $\max_{x \in [0, 1]} f(x)$ have for all such functions?

42.10.4. The union of several discs has an area of 1. Prove that it is possible to find several non-intersecting discs among them with the total area $> \frac{1}{9}$.

42.10.5*. See Problem 42.9.5.

Olympiad 43 (1980)*Grade 7*

43.7.1. Find the greatest five-digit number A in which the fourth digit is greater than the fifth; the third greater than the sum of the fourth and fifth; the second greater than the sum of the third, fourth and fifth; and the first greater than the sum of the other digits.

43.7.2. In every square of a rectangular graph paper stands 1 or -1 . The number of 1's is not less than two and the number of -1 's is not less than two. Prove that there are two rows and two columns such that the sum of the four numbers in the squares at their intersections is equal to 0.

43.7.3. Consider a convex 100-gon. Prove that the greatest number of sides of a convex polygon, whose sides lie on diagonals of the 100-gon, is ≤ 100 .

43.7.4. Three straight corridors of equal length l form a figure shown in Fig. 88. A cop and a robber are running along the corridors. The top speed of the cop is two times that of the robber. The cop is shortsighted and can only recognize the robber when the distance between them is $\leq r$. Prove that the cop will always catch the robber if a) $r > \frac{l}{3}$; b) $r > \frac{l}{4}$. (See Problem 437.4.)

FIGURE 88. (Probl. 43.7.4)

43.7.5. Ten vertices of a regular 20-gon $A_1A_2A_3 \dots A_{20}$ are painted black, and 10 are painted white. Consider the set consisting of diagonal A_1A_4 and all the other diagonals of the same length. Prove that in this set the number of diagonals with two black endpoints is equal to the number of diagonals with two white endpoints.

Grade 8

43.8.1. Prove that if $a_1 \leq a_2 \leq a_3 \leq \dots \leq a_{10}$, then

$$\frac{a_1 + \dots + a_6}{6} \leq \frac{a_1 + \dots + a_{10}}{10}.$$

43.8.2. See Problem 43.7.2.

43.8.3. * A point C is on a chord AB of circle K with center at O . Let D be the second intersection point of K with the circle circumscribed around $\triangle ACO$. Prove that $CD = CB$.

43.8.4. See Problem 43.7.4.

43.8.5. See Problem 43.7.5.

Grade 9

43.9.1. Let $a_1 < a_2 < a_3 < \dots$ be an increasing sequence of positive integers such that $a_{n+1} \leq 10a_n$ for any $n \in \mathbb{N}$. Prove that the infinite decimal fraction $0.a_1a_2a_3\dots$ obtained by writing these numbers one after another is non-periodic.

43.9.2. There are several push buttons on a panel that controls lamps on a desk. Pressing any button turns some lamps on the desk on or off (every button governs its own set of lamps and the sets may intersect). Prove that the number of all possible states of the desk is equal to a power of 2.

43.9.3. On an $m \times n$ rectangular piece of graph paper there are several squares whose sides are on the vertical and horizontal lines of the paper. It is known that no two squares coincide and no square is situated inside another one. What is the maximal number of such squares?

43.9.4. See Problem 43.7.4 for a) $r > \frac{l}{5}$; b*) $r > \frac{l}{7}$.

43.9.5*. See Problem 43.8.3.

Grade 10

43.10.1. See Problem 43.9.1.

43.10.2. See Problem 43.9.2.

43.10.3. See Problem 43.9.4.

43.10.4. One of the numbers $-1, 0$ or 1 is written in every square of a 1980×1980 table. The sum of all numbers is equal to 0. Prove that there exist two rows and two columns such that the sum of the four numbers written in the squares of their intersections is equal to 0.

43.10.5. On a unit sphere, there are given several arcs of great circles. The sum of the length of all these arcs is less than π . Prove that there is a plane passing through the center of the sphere and not intersecting any of the arcs; see Fig. 89.

FIGURE 89. (Probl. 43.10.5)

Olympiad 44 (1981)

Grade 7

44.7.1. The remainders after divisions of a positive integer A by 1981 and 1982 are both equal to 35. What is the remainder after division of A by 14?

44.7.2. See Problem 44.9.1 below for a 13-digit number.

44.7.3. A painter drew two identical dragons on two identical paper discs so that the first dragon's eye is at the center of the first disc and the second dragon's eye is not at the center of the second disc. Prove that it is possible to cut the second disc into two parts so that they can be put together again and the same disc with the same dragon can be obtained but this time the dragon's eye will be at the center.

44.7.4. Recall that $[x]$ is the integer part of x . For a number x greater than 1, is it necessary that $\left[\sqrt{\left[\sqrt{x}\right]}\right] = \left[\sqrt{\sqrt{x}}\right]$?

44.7.5. There are 5 identically looking weights. Their masses are 1000 g, 1001 g, 1002 g, 1004 g, and 1007 g but we do not know which mass is which. Given a balance with an arrow that shows mass in grams, how to find the weight with mass 1000 g in three weighings?

Grade 8

44.8.1. In a pentagon, all diagonals are drawn. Which 7 angles between the diagonals or between the diagonals and the sides should be marked so that if the angles marked are equal it would follow that the pentagon is regular?

44.8.2. See Problem 44.7.2.

44.8.3. See Problem 44.7.3.

44.8.4. See Problem 44.7.4.

44.8.5. Given 10 positive integers $a_1 < a_2 < a_3 < \dots < a_{10}$, prove that their least common multiple is not less than $10a_1$.

Grade 9

44.9.1. A number is expressed with an odd number of digits. Prove that it is possible to strike out one of its digits so that in the number obtained, there are as many 7's in even places as in odd places.

44.9.2. Positive integers a_1, a_2, \dots, a_n are such that each of them is not greater than its index (i.e., $a_k \leq k$), and the sum of all numbers is even. Prove that one of the sums $a_1 \pm a_2 \pm a_3 \pm \dots \pm a_n$ is equal to zero.

44.9.3. X and Y are two convex polygons, X lies inside Y . Let $S(X)$ and $S(Y)$ be the areas of the polygons, and $P(X)$ and $P(Y)$ be their perimeters. Prove that $\frac{S(X)}{P(X)} < 2\frac{S(Y)}{P(Y)}$.

44.9.4* Is it possible to divide the set of positive integers into an infinite number of infinite subsets, so that each subset can be obtained from any other one by adding a fixed integer element-wise?

44.9.5* 64 vertices of a regular 1981-gon are marked. Prove that there exists a trapezoid with vertices in marked points.

Grade 10

44.10.1. A function $y = f(x)$ is defined on the whole real line and satisfies the relation $f(x+k)(1-f(x)) = 1+f(x)$ for some $k \neq 0$. Prove that $f(x)$ is a periodic function.

44.10.2. Given a positive integer p and a polynomial $P(x)$ of degree n with leading coefficient 1 and such that if y is an integer, then $P(y)$ is an integer divisible by p . Prove that $n!$ is divisible by p . (Cf. Problems 20.1.7.2 and 20.1.8.5.)

44.10.3. Prove that the sequence $x_n = \sin(n^2)$ does not tend to 0 as $n \rightarrow \infty$.

44.10.4. Inside a unit square lies a non-selfintersecting broken line of length ≥ 200 . Prove that there is a straight line parallel to one of the sides of the square that intersects the broken line in no fewer than 101 points.

44.10.5. Consider a triangle. The radius of the inscribed circle is equal to $\frac{4}{3}$; the lengths of the triangle's heights are integers whose sum is equal to 13. Find the lengths of the triangle's sides.

44.10.6*. n people sit at a round table. Any two neighbors may change places. What is the least number of times that people must change places so that in the end they all have their initial neighbors but in the reverse order?

Olympiad 45 (1982)

Grade 7

45.7.1. At *Turing Machines* store Pete bought a calculator that performs the following operations: it can calculate $x + y$ and $x - y$ for any numbers x and y and $\frac{1}{x}$ for $x \neq 0$. Pete says that he can find the square of any positive number in not more than 6 operations on his calculator. a) If you also can, explain how. b) Can you, moreover, multiply any two positive integers in not more than 20 operations if you are allowed to write down intermediate results and use them during your calculations many times?

45.7.2. There are 5 points inside square $ABCD$. Prove that the distance between some two of them is not greater than $\frac{AC}{2}$.

45.7.3. At *Turing Machines* store Pete bought a paid calculating machine that for 5 kopeks multiplies any number punched into it by 3 and for 2 kopeks adds 4 to any number. Pete wants to obtain the number 1981 for the least amount of money and begins with 1 which may be punched in for free. How much will his calculations cost Pete's parents? Same question if he wants to obtain 1982.

45.7.4. What least number of points on a plane must be selected so that among all distances between pairs of points there should be 1, 2, 4, 8, 16, 32, 64?

Grade 8

45.8.1*. Simplify the expression:

$$\frac{2}{\sqrt{4 - 3\sqrt[4]{5} + 2\sqrt{5} - \sqrt[4]{125}}}.$$

45.8.2. A rectangle is cut into 5 rectangles. Prove that there is a pair of these 5 rectangles one of which fits completely inside the other.

45.8.3. The squares of 1, 2, 3, ..., 1982 are juxtaposed in some order to form a number. Can the number obtained be the square of an integer?

45.8.4. All diagonals of a convex pentagon are parallel to the opposite sides. Prove that the ratio of every diagonal to the opposite side is equal to $\frac{\sqrt{5}+1}{2}$.

45.8.5. a) Knowing that (one can easily prove this by induction)

$$1^3 + 2^3 + \dots + n^3 = \left(\frac{n(n+1)}{2}\right)^2,$$

prove that for distinct positive integers a_1, a_2, \dots, a_n the following inequality holds:

$$(a_1^7 + a_2^7 + \dots + a_n^7) + (a_1^5 + a_2^5 + \dots + a_n^5) \geq 2(a_1^3 + a_2^3 + \dots + a_n^3)^2.$$

b) Are there some distinct positive integers a_1, a_2, \dots, a_n for which the equality is attained?

Grade 9

45.9.1. Find all integers n for which the number $n2^n + 1$ is divisible by 3.

45.9.2. On a plane find a point such that the sum of the distances from it to four given points is minimal.

45.9.3. On a plane the points with integral coordinates are marked. Prove that there exists a circle with exactly 1982 marked points inside it.

45.9.4. The number

$$A = 0.1 + 0.02 + 0.003 + \dots + n \cdot 10^{-n} + \dots$$

is written in the form of an infinite decimal fraction. Prove that the digits 1982 in succession do not appear in this decimal.

45.9.5. Two sides of a convex quadrilateral are of length 1 and two other sides and both diagonals are not longer than 1. What is the longest possible perimeter of the quadrilateral?

Grade 10

45.10.1. a) Prove that if all edges of a regular tetrahedron subtend equal angles with a common vertex inside the tetrahedron, then this vertex is the center of the sphere circumscribed around the tetrahedron.

b) Can the vertices of equal angles subtending the tetrahedron's edges be outside the tetrahedron?

Note: If the vertex lies on an edge or its extension, we say that the edge subtends an angle of π or 0, respectively.

45.10.2. a) Let a, b, c be the lengths of a triangle's sides. Prove that

$$a^4 + b^4 + c^4 - 2(a^2b^2 + a^2c^2 + b^2c^2) + a^2bc + b^2ac + c^2ab \geq 0.$$

b) Prove that the inequality in a) holds for any $a, b, c \geq 0$.

45.10.3. Pete bought a useful calculator at the *Turing Machines* store: it can find $xy + x + y + 1$ for any real numbers x and y but cannot perform any other operations. Pete wants to write a "program" to compute the polynomial $1 + x + x^2 + \dots + x^{1982}$. He regards his "program" to be the sequence of polynomials $f_1(x), \dots, f_n(x)$ such that

$$f_1(x) = x; \quad f_n(x) = 1 + x + \dots + x^{1982};$$

$f_i(x)$ for $1 < i < n$ is either a constant c_i that Pete can choose or $f_j(x) \cdot f_k(x) + f_k(x) + f_j(x) + 1$, where $j, k < i$ for each $i = 2, \dots, n$.

a) Write Pete's "program".

b) Can one write a "program" for the calculator that can only perform the following operation: $x, y \mapsto xy + x + y$?

45.10.4. Find all positive integers n for which both $\frac{1}{n}$ and $\frac{1}{n+1}$ are finite decimal fractions.

45.10.5. A regular hexagon with side a is inside another regular hexagon with side $2a$. Prove that the center of the larger hexagon is inside the smaller one.

Olympiad 46 (1983)

Grade 7

46.7.1. Find all pairs of integers (x, y) satisfying the equation

$$x^2 = y^2 + 2y + 13.$$

46.7.2. A white plane is stained with black Indian ink. Prove that for any l there exists a line segment of length l whose both endpoints are of the same color.

46.7.3. A positive integer begins with a 4. If this digit 4 is transplanted to the end of the number, the resulting number is $\frac{1}{4}$ of the original one. Find the smallest such number.

46.7.4. Two friends want to reach a nearby town. They have a bicycle for one person only. The Rule allows any of them to leave the bicycle for the other friend at any place. Their speeds as pedestrians are u_1 and u_2 , their speeds on bicycles are v_1 and v_2 , respectively, and the distance between the towns is S . What is the least time the friends need to reach the town?

46.7.5. Is there a pentagon with sides 3, 4, 9, 11 and 13 cm, into which a circle can be inscribed?

Grade 8

46.8.1. Prove that $x^4 - x^3y + x^2y^2 - xy^3 + y^4 > x^2 + y^2$ for any $x > \sqrt{2}$ and $y > \sqrt{2}$.

46.8.2. Equilateral triangles ABC_1, BCA_1 and CAB_1 are constructed outwards on the sides of triangle ABC . Prove that $\overrightarrow{AA_1} + \overrightarrow{BB_1} + \overrightarrow{CC_1} = \vec{0}$. (See Fig. 90.)

46.8.3. Can the square of a positive integer begin with 1983 nines in a row?

46.8.4. The numbers 1, 2, \dots , 1983 stand at the vertices of a regular 1983-gon. Any of the axes of symmetry of the 1983-gon divides the numbers which do not stand at the vertices through which the axis passes (if any) into two sets: on either side of the axis. Let us call an arrangement of numbers *good* with respect to a given axis of symmetry if every number of one set is greater than the number symmetrical to it. Is there an arrangement good with respect to any axis of symmetry?

46.8.5. Given five points on a circle: A_1, A_2, A_3, A_4, H . Denote the distance between H and straight line A_iA_j by h_{ij} . Prove that $h_{12} \cdot h_{34} = h_{14} \cdot h_{23}$.

FIGURE 90. (Probl. 46.8.2)

Grade 9

46.9.1. Prove that

$$\frac{1}{2} < x^{2n} \pm x^{2n-1} + x^{2n-2} \pm x^{2n-3} + \dots + x^4 \pm x^3 + x^2 \pm x + 1$$

for any signs of odd powers of a real x .

46.9.2. Three circles of radii 3, 4, 5 are externally tangent to one another. The common tangent to the first two circles is drawn through the point at which they are tangent to each other. Find the length of this tangent contained inside the circle of radius 5.

46.9.3. Prove that $1^{1983} + 2^{1983} + \dots + 1983^{1983}$ is divisible by $1 + \dots + 1983$.

46.9.4. Twenty towns are connected by 172 airlines; not more than one airline connects two towns. Prove that using these airlines one can fly from any town to any other (perhaps changing lines).

Grade 10

46.10.1. Let A_1, B_1, C_1 be the points where the circle inscribed into triangle ABC is tangent to sides BC, AC and AB , respectively. It is known that $AA_1 = BB_1 = CC_1$. Prove that triangle ABC is equilateral.

46.10.2. Prove that $4^m - 4^n \vdots 3^{k+1}$ if and only if $m - n \vdots 3^k$, where a) $k = 1, 2, 3$; b) $k \in \mathbb{N}$.

46.10.3. After classes, the following inscription was left on a blackboard (instead of the erased numbers we write *** in this book):

“Find

$$t(0) - t\left(\frac{\pi}{5}\right) + t\left(\frac{2\pi}{5}\right) - t\left(\frac{3\pi}{5}\right) + \dots + t\left(\frac{8\pi}{5}\right) - t\left(\frac{9\pi}{5}\right), \tag{*}$$

where

$$t(x) = \cos 5x + *** \cos 4x + *** \cos 3x + *** \cos 2x + *** \cos x + *** \cos 0 \tag{**}”$$

A student told his girlfriend that he could find the sum (*) even without knowing the coefficients erased from the blackboard in (**). Is he just boasting?

46.10.4. Consider eight points in space such that no four of them lie on the same plane, and 17 segments with both endpoints in given points. Prove that the segments form a) at least one triangle; b) ≥ 4 triangles.

46.10.5. 13 knights from k towns ($1 < k < 13$) are sitting at a round table. Every knight holds a gold or a silver goblet in his hand, and the number of gold goblets is also equal to k . Prince tells every knight to pass his goblet to the neighbor on his right and to repeat this *until a pair of knights from the same town gets golden goblets*. Prove that eventually Prince’s wish will be fulfilled and the knights will be able to pass to refreshments.

Olympiad 47 (1984)

Grade 7

47.7.1. Some people call a bus ticket *lucky* if the sum of digits in its number is divisible by 7. Is it possible for two tickets with consecutive numbers to be lucky?

Note. In 1984 bus tickets in Moscow were numbered 000000 to 999999.

47.7.2. Paths in a zoo form an equilateral triangle with the midpoints of its sides connected. A monkey has run away from its cage and two guards are trying to catch it. Can they catch the monkey if all three run only along the paths, the speed of the monkey and that of the guards are equal and they all can see one another at all times? (Cf. Problem 33.2.8.2).

47.7.3. A customer bought some goods worth 10 roubles and gave a 25-rouble note to the salesman. The salesman did not have change at the moment and so he asked his neighbor to change the note. After they got even and the customer had gone, the neighbor discovered that the note was counterfeit. The salesman returned 25 roubles to his neighbor and pondered: how much money did he lose? Same question to you.

47.7.4. A parallelogram is cut out of a paper triangle. Prove that the area of the parallelogram is not greater than half the area of the triangle. (See Fig. 91.)

FIGURE 91. (Probl. 47.7.4)

47.7.5. There are 10 rooks and a king on a 20×20 chessboard. The king is not in check and moves along the diagonal from the lower left corner to the upper right corner. The pieces move taking turns as follows: first the king, then one of the rooks.

Prove that no matter what the initial position of the rooks is or how they move, the king will either be in check or bump into a rook.

Grade 8

47.8.1. Solve the equation $\frac{x^3}{\sqrt{4-x^2}} + x^2 - 4 = 0$.

47.8.2. Every two of six computers are to be connected by one colored cable. Choose one color out of five for each cable so that cables of five different colors would come out of each computer.

47.8.3. Prove that the sum of distances from the center of a regular heptagon to all its vertices is less than that from any other point.

47.8.4. The sum of five non-negative numbers is equal to 1. Prove that it is possible to arrange them in a circle so that the sum of all five products of pairs of neighboring numbers is not greater than $\frac{1}{5}$.

47.8.5. Cut a square into 8 acute triangles. (Cf. Problem 47.10.5 below.)

47.8.6. Is the number of all 64-digit positive integers without zeros in their decimal expression and are divisible by 101 even or is it odd?

Grade 9

47.9.1. In a triangular pyramid 3 lateral edges are equal to one another, and the areas of three lateral faces are equal to one another. Prove that the base of the pyramid is an isosceles triangle.

47.9.2. Is it possible to connect 13 computers in pairs with cables of twelve different colors so that 12 cables of different colors come out of each computer? (Cf. Problem 47.8.2.)

47.9.3. What is the least possible width of an infinite strip from which any triangle of area 1 can be cut out?

47.9.4. On a circle, there are arranged n non-negative numbers whose sum is equal to 1. Prove that the sum S_n of n products of two neighboring numbers is not greater than $\frac{1}{4}$. (Cf. Problem 47.8.4).

47.9.5*. Given 4 points inside a 3×4 rectangle. Prove that there are two among the given points that are not farther than $\frac{25}{8}$ apart.

47.9.6. Do there exist three non-zero digits with which the squares of an infinite number of different integers can be expressed?

Grade 10

47.10.1. Prove (without using calculators, tables and such) that $\sin 1 < \log_3 \sqrt{7}$.

47.10.2. At the Olympiad 6 problems were offered. The Olympiad jury decided to assign to every participant a positive integer according to his/her results in the Olympiad so that it would be possible to reconstruct unambiguously the score every participant got for every problem and so that for every two participants the greater number would be assigned to the one with the greater sum of scores. How could the jury enumerate the participants?

47.10.3. Solve in integers $19x^3 - 84y^2 = 1984$.

47.10.4. Let $n_1 < n_2 < n_3 < n_4 < \dots$ be an infinite sequence of positive integers. In a kingdom there was minted an infinite number of coins of denominations $n_1, n_2, n_3, n_4, \dots$ kopeks.

Prove that it is possible to break the sequence at some point N so that any amount of money which can be paid without need for change with all coins minted can in fact be paid with the coins of denominations of $n_1, n_2, n_3, \dots, n_N$ kopeks only.

47.10.5. A square is cut into acute triangles. Prove that there are ≥ 8 such triangles. (Cf. Problem 47.8.5).

47.10.6*. A triangle section of a cube is tangent to the sphere inscribed in the cube. Prove that the area of the section is less than half the area of the cube's face.

Olympiad 48 (1985)

Grade 7

48.7.1. Solve the equation $xy + 1 = x + y$.

48.7.2. Given five distinct positive numbers. They can be divided into two groups so that the sums of the numbers in these groups are equal. In how many ways can this be done?

48.7.3. The lengths a, b, c, d of four segments satisfy the inequalities $0 < a \leq b \leq c < d$ and $d < a + b + c$. Is it possible to construct a trapezoidal from these segments?

48.7.4. A rabbit is sitting in the center of a square and 4 wolves are sitting in the four vertices. Is it possible for the rabbit to run out of the square if the wolves can only run along the sides and the wolf's top speed is 1.4 times higher than that of the rabbit?

48.7.5. A tank of milk was brought to a store. The salesman has a balance and pans but no weights. However, milk cans can be put on a pan and there are three identical milk cans in the store, two of which are empty, and the third one has 1 liter of milk in it. A can holds not more than 85 l. By a *weighing* we mean putting a can with milk on one balance pan and an empty can on the other pan whereupon milk is added to the empty can until the balance is in equilibrium. How can the salesman pour 85 l of milk into one can weighing not more than 8 times?

Grade 8

48.8.1. Solve the equation $(x - y + z)^2 = x^2 - y^2 + z^2$.

48.8.2. The numbers $a_1, a_2, \dots, a_{1985}$ are the numbers $1, 2, 3, \dots, 1985$ arranged in some order. Prove that $\max_k k \cdot a_k \geq 993^2$.

48.8.3. A paper square Q is placed on a piece P of graph paper; the area of Q is four times that of a little square q of the graph paper. Let a *node* be an intersection of lines on the paper; a node on the boundary of Q is considered to be *covered*. What is the least number of nodes that Q can cover? (See Fig. 92.)

48.8.4. An infinite number of knights lined up in a row in front of Wizard. Prove that Wizard can tell some of them to stand out of line, so that there would still be an infinite number of knights left in line, and so that all knights in line would stand ordered with respect to their height in increasing or decreasing order.

48.8.5. Prove that if the length of every one of the three bisectors of a triangle is greater than 1, then its area is greater than $\frac{1}{\sqrt{3}}$.

Grade 9

48.9.1. Solve the equation $\sqrt{x - y + z} = \sqrt{x} - \sqrt{y} + \sqrt{z}$.

FIGURE 92. (Probl. 48.8.3)

48.9.2. In some country there are 1985 airports. Consider the Earth to be a plane, the air routes to be straight lines, and all pairs of distances between the airports to be distinct. From every airport an airplane departs and lands at the airport farthest from the place of its departure. Is it possible that as a result all 1985 airplanes arrived in 50 airports?

48.9.3. Under notations of Problem 48.8.3, suppose we know that a 2×2 square covers ≥ 7 nodes of the graph plane. How many nodes (exactly) can a 2×2 square cover?

48.9.4. Prove that it is possible to select two people from a group of 12, and then choose five more people from the remaining 10 so that each of these five people satisfies the following condition: (s)he is either a friend of both or of neither of the people in the pair chosen first.

48.9.5* (*Leonard Euler's problem*). . Prove that any number 2^n for $n \geq 3$ can be expressed as $2^n = 7x^2 + y^2$, where x and y are odd.

Grade 10

48.10.1. Solve the equation

$$\frac{x-49}{50} + \frac{x-50}{49} = \frac{49}{x-50} + \frac{50}{x-49}.$$

48.10.2. See Problem 48.7.3.

48.10.3. Let the “complexity” of a given number be the least possible length of a numerical sequence (if there is one) which begins with a 0 and ends with this number, each next term being either equal to half the preceding one or its sum with the preceding term being equal to 1. (The length of the empty sequence is assumed to be equal to 0.) Find the number with the greatest “complexity” among all numbers of the form $\frac{m}{2^{50}}$, where $m = 1, 3, 5, \dots, 2^{50} - 1$.

48.10.4. We have 1985 sets. Each of the sets has 45 elements, the union of any two sets has exactly 89 elements. How many elements has the union of all these 1985 sets?

48.10.5. Prove that if the distances between skew edges of a tetrahedron are equal to h_1, h_2, h_3 , respectively, then the volume of the tetrahedron is $\geq \frac{1}{3}h_1h_2h_3$.

Olympiad 49 (1986)

Grade 7

49.7.1. A quadrilateral is drawn on a transparent piece of paper. How should the paper be folded (perhaps more than once) in order to ascertain whether the quadrilateral is a rhombus? (Cf. Problem 49.8.1.)

49.7.2. Prove that there are no numbers x, y, z satisfying the system

$$\begin{cases} |x| < |y - z|, \\ |y| < |z - x|, \\ |z| < |x - y| \end{cases}$$

49.7.3. Three dwarfs live in different houses on a plane and walk with speeds 1, 2 and 3 km/h, respectively. What place for their everyday meetings should they choose to minimize the sum of the times it takes them to walk from their houses to this place (each walks along a straight line)?

49.7.4. The product of some 1986 positive integers has exactly 1985 different prime divisors. Prove that either one of these integers or the product of some of them is a perfect square. (Cf. Problem 49.9.4.)

49.7.5. A code lock has three buttons with numbers 1, 2, 3. The code is a three-digit number, and the lock opens only if you press all three buttons in succession in the right order. What least number of times must Houdini press the buttons to unlock the lock?

Grade 8

49.8.1. A quadrilateral is drawn on a transparent piece of paper. How should the paper be folded (perhaps more than once) in order to ascertain whether the quadrilateral is a square? (Cf. Problem 49.7.1.)

49.8.2. Find all positive integers which cannot be expressed as the difference of the squares of some positive integers.

49.8.3. Prove that if $a_1 = 1$, $a_n = \frac{a_{n-1}}{2} + \frac{1}{a_{n-1}}$ for $n = 2, 3, \dots, 10$, then $0 < a_{10} - \sqrt{2} < 10^{-370}$.

49.8.4. A square field is divided into 100 identical square plots, nine of which become overgrown with weeds. It is known that every next year weeds begin to grow on the plots which are adjacent (have a common side) to at least two plots overgrown with weeds the year before and only on these plots. Prove that the whole field will never become overgrown with weeds.

49.8.5. Prove that there are no solutions to the system

$$\begin{cases} |x| > |y - z + t|, \\ |y| > |x - z + t|, \\ |z| > |x - y + t|, \\ |t| > |x - y + z|. \end{cases}$$

Grade 9

49.9.1. Points A, B, C, D are marked on a piece of paper. A detecting device can perform two types of operations: (a) measure the distance between two given points in centimeters; (b) compare two given numbers. What least number of operations must be performed to ascertain whether quadrilateral $ABCD$ is a rectangle?

49.9.2. An ant moves at a constant speed starting from point M on a plane. Its path is a spiral that winds around a point O and is homothetic to some part of itself with respect to this point. Is it possible for the ant to cover its entire pass in a finite time?

49.9.3. Solve the system:

$$\begin{cases} |x| < |y - z + t|, \\ |y| < |x - z + t|, \\ |z| < |x - y + t|, \\ |t| < |x - y + z|. \end{cases}$$

49.9.4. A product of some 48 positive integers has exactly 10 different prime divisors. Prove that the product of some four of these integers is a perfect square. (Cf. Problem 49.7.4.)

49.9.5. Discs of radius $\frac{1}{14}$ and with centers at every point with integer coordinates are drawn on the coordinate plane. Prove that any circle of radius 100 intersects at least one of the discs drawn.

Grade 10

49.10.1. See Problem 49.9.1 with *rectangle* replaced with *square* in the question.

49.10.2. The bisector of angle A of triangle ABC is extended until it meets (at point D) the circumscribed circle. (See Fig. 93.) Prove that $AD > \frac{1}{2}(AB + AC)$.

49.10.3. Solve the equation $x^{x^4} = 4$ for $x > 0$.

49.10.4. Prove that there are no vector solution to the system:

$$\begin{cases} \sqrt{3}|\mathbf{a}| < |\mathbf{b} - \mathbf{c}|, \\ \sqrt{3}|\mathbf{b}| < |\mathbf{c} - \mathbf{a}|, \\ \sqrt{3}|\mathbf{c}| < |\mathbf{a} - \mathbf{b}|. \end{cases}$$

49.10.5. For $y(x) = |\cos x + \alpha \cos 2x + \beta \cos 3x|$ find $\min_{\alpha, \beta} \max_x y(x)$.

FIGURE 93. (Probl. 49.10.2)

Olympiad 50 (1987)

Grade 7

50.7.1. In March the math club held 11 meetings. Prove that if there were no meetings on weekends, then in March there were three days in a row during which no meetings were held.

50.7.2. Prove that among any 27 different positive integers less than 100 each there are two not relatively prime ones.

50.7.3. On a meadow shaped in the form of an equilateral triangle with side 100 m a wolf is running. A hunter can hit the wolf if (s)he shoots from a distance not greater than 30 m. Prove that the hunter can hit the wolf no matter how quickly it runs.

FIGURE 94. (Probl. 50.7.3)

FIGURE 95. (Probl. 50.8.4)

50.7.4. Let AB be the base of trapezoid $ABCD$. Prove that if $AC + BC = AD + BD$ then $ABCD$ is an isosceles trapezoid.

50.7.5. Ali-Baba and 40 thieves have to split a treasure of 1987 gold coins among themselves according to the following Rule: the first thief splits the whole treasure into two parts; then the second thief divides one of these parts into two parts, etc. After the fortieth division, the first thief takes the greatest of the parts; then the second thief takes the greatest of the remaining parts, etc. The last, fortyfirst, part goes to Ali-Baba.

What is the greatest number of coins each thief can get under this Rule regardless of the other thieves' actions?

Grade 8

50.8.1. Prove that $\frac{1}{2} \left(\frac{x}{a} + \frac{y}{b} \right) > \frac{x+y}{a+b}$ for $a > b > 0$ and $\frac{x}{a} < \frac{y}{b}$.

50.8.2. A boy decided to cut out of a $2n \times 2n$ piece of paper the greatest possible number of $1 \times (n+1)$ rectangles. What is this number if: a) $n < 3$; b) $n = 3$; c) $n > 3$?

50.8.3. A teacher organizes a tug-of-war tournament and decides that all possible teams that can be made from students of her class (obviously not counting the whole class as a team) should participate exactly once. Prove that each team will compete with the team made up of the remaining students.

50.8.4. In pentagon $ABCDE$, $\angle ABC$ and $\angle CDE$ are right angles, $\angle BCA = \angle DCE$, and M is the midpoint of side AE . Prove that $MB = MD$ (See Fig. 95.)

50.8.5. Is there a set of positive integers such that for any positive integer n at least one of the numbers n , $n + 50$ belongs to the set, and at least one of the numbers n or $n + 1987$ does not?

Grade 9

50.9.1. Given a set of 7 different integers from 0 to 9. Prove that for any positive integer n there exists a pair of integers from the set whose sum ends with the same digit as n does.

50.9.2. Given k vertices of a regular pentagon, find the remaining vertices using a two-sided ruler for a) $k = 4$, b) $k = 3$.

50.9.3. Find 50 positive integers such that none of them is divisible by another, and the product of any two is divisible by any of the rest.

50.9.4. Prove that if $n = 1987$, then

$$\frac{(a_1 + \cdots + a_n)^2}{b_1 + \cdots + b_n} \leq \frac{a_1^2}{b_1} + \cdots + \frac{a_n^2}{b_n}$$

for any a_1, a_2, \dots, a_n and positive b_1, b_2, \dots, b_n .

50.9.5. Tanya dropped a ball into a huge rectangular pool. She wants to rescue it using 30 narrow planks, each 1 m long to make a bridge so that each plank is supported by either the edges of the pool or by the planks already settled, and so that ultimately one of the planks is right over the ball. Prove that Tanya will not be able to do this if the distance from the sides of the pool to the ball exceeds 2 m. (See Fig. 96.)

FIGURE 96. (Probl. 50.9.5)

Grade 10

50.10.1. a) Prove that of three positive numbers it is always possible to select two, say, x and y , so that $0 \leq \frac{x-y}{1+xy} \leq 1$.

b) Is it possible to select such numbers from any 4 (not necessarily positive) numbers?

50.10.2. The measures of the angles between a plane in space and the sides of an equilateral spatial triangle are equal to α, β, γ . Prove that one of the numbers $\sin \alpha, \sin \beta, \sin \gamma$ is equal to the sum of the other two.

50.10.3. On a piece of graph paper, 17 squares with side 1 are shaded. Prove that they can be covered by rectangles, the sum of whose perimeters is less than 100, so that the distance between any two points on distinct rectangles is $\geq \sqrt{2}$.

50.10.4. Is it possible to divide the set of integers into 3 subsets so that for any integer n the numbers $n, n - 50, n + 1987$ would belong to different subsets?

50.10.5. The side of a square shaped kingdom is 2 km. The king of this kingdom decides to summon all his subjects to a ball at 7 p.m. At noon he sends a messenger who may give any orders to any citizen who, in turn, is empowered to give any order to any other citizen, etc. The whereabouts (home) of each citizen are known and every citizen can move at a speed of 3 km/h in any direction. Prove that the king can organize the transmission of messages so that all his loyal subjects can reach the court in time for the opening of the ball.

Olympiad 51 (1988)

Grade 7

51.7.1. Prove that for any prime $p > 7$ the number $p^4 - 1$ is divisible by 240.

51.7.2. Points M and P are the midpoints of two edges of a cube. On the surface of the cube, find the locus of points equidistant from M and P . The distance between two points of the surface is calculated as the length of the shortest broken line lying on the surface.

51.7.3. Using only a ruler and calipers draw the straight line through a given point and parallel to a given line.

51.7.4. Colored wires connect 20 phones so that each wire connects two phones, not more than one wire connects each pair of phones and not more than two wires lead from each phone. By the Rule we should select the colors of the wires so that every two wires leading from the same phone have different colors. What is the least number of wire's colors needed for such a connection? (Cf. Problem 51.9.5.)

Grade 8

51.8.1. Four numbers: 1, 9, 8, 8 are written in line. We apply to them the following operation: between each two numbers a and b we write their difference $b - a$. Then the same operation is applied to the resulting line, and so on, 100 times. What is the sum of all numbers in the final line?

51.8.2. Find the midpoint of a given segment using only a ruler without marks on it and calipers.

51.8.3. Prove that the equation $3x^4 + 5y^4 + 7z^4 = 11t^4$ has no solution in natural numbers.

51.8.4. There are four coins and a spring balance with a single pan. It is known that some of the coins may be forged and a real coin weighs 10 g while a forged one only 9 g. How many times has one to weigh the coins to find out for sure which of them are forged?

Grade 9

51.9.1. Consider a convex quadrilateral. Its diagonals divide it into four triangles of integer area. Prove that the product of these four integers cannot end with digits 1988.

51.9.2. Prove that $p_1^2 + p_2^2 + \dots + p_{24}^2 \div 24$ for any primes $p_1, p_2, \dots, p_{24} \geq 5$.

51.9.3. Two perpendicular straight lines lie on a plane. Using only calipers find three points on the plane that represent vertices of an equilateral triangle.

51.9.4. Let $f(x, y) = \frac{1}{2}(x + y - 1)(x + y - 2)$ be a function of two positive integers. Prove that for any positive integer z there exists a single pair x, y such that $f(x, y) = z$.

51.9.5. Colored wires connect 20 phones so that each wire connects two phones, not more than one wire connects each pair of phones and not more than three wires lead from each phone. One is asked to select the colors of the wires so that every two wires leading from the same phone have different colors. What is the least number of wires' colors needed to establish any such connection?

Grade 10

51.10.1. A calculator can add, subtract, divide, multiply and take the square root. Find a formula to calculate the minimum of two numbers using the calculator.

51.10.2. Is there a straight line on the coordinate plane such that the graph of the function $y = 2^x$ is symmetric with respect to this line?

51.10.3. Can one intersect any parallelepiped with a plane so that the section is a rectangle?

51.10.4. One has a one-sided ruler, a pencil and a length standard allowing one to find on a previously drawn straight line a point at fixed distance from some other point on the same line. Draw a perpendicular to a given straight line using only these instruments.

51.10.5. One selects a pair of positive integers and performs the following *operation*: the greater number of the pair (the first one if they are equal) is divided by the other number, and the pair: (the quotient, the remainder) replace the original pair. Then the operation is repeated until the smaller number becomes 0. We start with numbers not greater than 1988. Prove that not more than 6 operations can be performed.

Olympiad 52 (1989)

Grade 7

52.7.1. We cut a square into 16 smaller equal squares. How to place each of the letters $A, B, C,$ and D in the squares in four ways so that no horizontal, no vertical and none of the two greater diagonals would contain the same letters.

52.7.2. Given a fixed line l and passing through a given point not on l . With the help of a ruler and compass draw a straight line parallel to l and passing through the given point by drawing the least possible number of curves (circles and straight lines).

52.7.3. There are 4 pairs of socks of two different sizes and of two colors lying pell-mell on a shelf in a dark room. What is the minimal number of socks from the shelf that we should put into a bag, without leaving the room for inventory of the bag, in order to have in the bag two socks of the same size and color?

52.7.4. A tourist left a tourist lounge in a boat at 10:15. (S)he promised to come back not later than at 1:00 p.m. the same day. The speed of the river's current is known to be 1.4 km/h and the top speed of the boat in still water is 3 km/h. What is the greatest distance from the lounge that the tourist can cover if (s)he rests for 15 minutes after every 30 minutes of rowing without mooring and may turn back only after a rest?

52.7.5. Find all positive integers x satisfying the following condition: the product of the digits of x is equal to $44x - 86868$ and their sum is equal to a cube of a positive integer.

Grade 8

52.8.1. Solve the equation $(x^2 + x)^2 + \sqrt{x^2 - 1} = 0$.

52.8.2. Some randomly chosen squares of an infinite graph paper are red and the rest are white. A grasshopper jumps on red squares and a flea on white ones and each jump can be made over any distance vertically or horizontally. Prove that the grasshopper and the flea can find themselves side by side after at most three jumps.

52.8.3. Construct with the help of a ruler and compass the perpendicular to the given straight line passing through the given point (a) not in this line and (b) on this line. You may only draw the least possible number of curves (circles and straight lines).

52.8.4. A subset X of the set of all two-digit "numbers" 00, 01, ..., 98, 99 is such that any infinite sequence of digits contains two neighboring digits that form a number from X . What is the least cardinality of X ?

52.8.5. Prove that a party of scouts can be always divided into two teams so that the cardinality of the set of pairs of friends in the same team is less than that of the set of pairs of friends who found themselves in distinct teams.

52.8.6. If $|ax^2 + bx + c| \leq 1$ for $x \in [0, 1]$ what can the greatest possible value of $|a| + |b| + |c|$ be?

Grade 9

52.9.1. There are 4 different straight lines in space. Two lines are red and two are blue, any red line is perpendicular to any blue line. Prove that either red lines are parallel or blue lines are parallel.

52.9.2. Points $M, K,$ and L are selected on sides $AB, BC,$ and $AC,$ respectively, of $\triangle ABC$ so that $MK \parallel AC$ and $ML \parallel BC$. Segment BL meets MK at P while AK meets ML at Q . Prove that segments $PQ \parallel AB$.

52.9.3. The numbers A_1, A_2, \dots form a geometric progression, and so do B_1, B_2, \dots . We form a new sequence by adding the progressions term-wise: $A_1 + B_1, A_2 + B_2, \dots$, etc. Can you determine the fifth term of the new sequence if you know the first four of its terms?

52.9.4. The streets of a city are represented on a map as straight lines that divide a square into 25 smaller squares of side 1. (The borderline of the city is considered to be the union of 4 streets.) There is a snow plow at the bottom right corner of the bottom left square. Find the length of the shortest path for the plow to pass through all streets and come back to its starting point.

52.9.5. Find all positive numbers x_1, x_2, \dots, x_n that satisfy the system of n equations:

$$(x_1 + x_2 + \dots + x_k)(x_k + x_{k+1} + \dots + x_n) = 1, \quad k = 1, 2, \dots, n$$

if a) $n = 3,$ b) $n = 4,$ c) $n = 10,$ d) n is an arbitrary integer.

Grade 10

52.10.1. Solve the equation $\lg(x - 2) = 2x - x^2 + 3$.

52.10.2. Is there a function whose graph on the coordinate plane has a common point with any straight line?

52.10.3. Is it possible to put down crosses and noughts on a sheet of graph paper of an arbitrary (or infinite) size so that no three signs in a row would be the same on any vertical, horizontal, or diagonal line?

52.10.4. Consider n distinct natural numbers. Prove that any infinite arithmetic progression whose first term does not exceed its difference, d , contains 3 or 4 of the numbers considered if a) $n = 5$, b) $n = 1989$.

52.10.5. Calculate with an accuracy to 2.0 the least total length of the cuts that must be made to recut a unit square into a rectangle with diagonal of length 100.

52.10.6. We select a point on every edge of an arbitrary tetrahedron. We draw a plane through every three points that belong to edges with a common vertex. Prove that if three of the four planes thus drawn are tangent to the sphere inscribed into the tetrahedron, the fourth plane is also tangent to it.

Olympiad 53 (1990)

Grade 8

53.8.1. Prove that if $0 < a_1 < a_2 < \dots < a_9$, then

$$\frac{a_1 + a_2 + \dots + a_9}{a_3 + a_6 + a_9} < 3.$$

53.8.2. Let $M = m(n+9)(m+2n^2+3)$. What is the least number of distinct prime divisors the number M can have?

53.8.3. 11 winners of grades 8, 9, 10 and 11 were invited to pass a selection test to an Olympiad. Can they be arranged at a round table so that among any five successive students there are representatives of all four grades?

53.8.4. Quadrilateral $ABCD$ is inscribed in a circle; $AB = BC$. Let diagonals meet at O , let E be the other intersection point of CD with the circle that passes through B, C and O . Prove that $AD = DE$.

53.8.5. A display board composed of 64 bulbs is controlled by 64 buttons, each bulb being switched on/off by a separate button. Any set of buttons can be pushed simultaneously. This was done and the bulbs that lighted as a result were marked. What is the least number of switchings that allows one to find out which button controls which bulb?

Grade 9

53.9.1. 7 boys got together and each of them has three brothers among the other present. Prove that all seven boys are brothers.

53.9.2. Prove that among any 53 distinct natural numbers whose sum does not exceed 1990 there are two numbers whose sum is equal to 53.

53.9.3. Inside a circle of radius 1 point A is marked. We drew various chords through A and then drew a circle of radius 2 through the endpoints of each chord. Prove that all such circles for various points A are tangent to a certain fixed circle.

53.9.4. There are two counterfeit coins among 8 coins that look alike. One of the counterfeits is lighter and the other is heavier than a genuine coin. Can one find out in three weighings on scales without weights whether the two counterfeit coins together are heavier, lighter or of the same weight as two genuine coins?

53.9.5. The decimal representation of a rational number A is a periodic fraction with the period of length n . What is the longest length of the period of A^2 as A varies?

Grade 10

53.10.1. Can one cut a square into three pairwise non-equal and pairwise similar rectangles?

53.10.2. Find all primes p, q, r that satisfy $p^q + q^p = r$.

53.10.3. Prove that for all values of parameters a, b, c there is a number x such that

$$a \cos x + b \cos 3x + c \cos 9x \geq \frac{1}{2}(|a| + |b| + |c|).$$

53.10.4. How should four points in a disc be arranged so as to have the greatest product of all pairwise distances between them?

53.10.5. Points A, B, C, D in space are positioned so that segment BD subtends angles $\angle A$ and $\angle C$ of measure α and AC subtends angles $\angle B$ and $\angle D$ of measure β . Find the ratio $AC : BD$ if $AB \neq CD$.

Grade 11

53.11.1. Find $\max_{x,y}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$.

53.11.2. Prove that if a function $f(x)$ is continuous on $[0, 1]$ and satisfies the identity $f(f(x)) = x^2$ for all x , then $x^2 < f(x) < x$ for any $x \in (0, 1)$. Give an example of such a function.

53.11.3. In triangle ABC , consider median BD and bisector BE . Can it happen that BD is a bisector in $\triangle ABE$ and BE a median in $\triangle BCD$?

53.11.4. Prove that there is a multiple of any odd n , whose decimal representation contains only odd digits.

53.11.5. Four points are projections of a point to four faces of a tetrahedron. How are the points arranged in space?

Olympiad 54 (1991)

Grade 8

54.8.1. Prove that if $a > b > c$, then $a^2(b - c) + b^2(c - a) + c^2(a - b) > 0$.

54.8.2. Given points A and B on a plane, construct a point C on ray AB , such that $AC = 2AB$. Is it possible to do it using a compass with a fixed span r if a) $AB < 2r$, b) $AB \geq 2r$?

54.8.3. To guard a military installation around the clock, a day shift and a night shift are required. A sentry guard may take either a day, or a night shift, or work around the clock. In these cases the guard is given a leave of absence of not less than 1, 1.5 or 2.5 full days, respectively. What is the least number of guards necessary to ensure the security of the installation?

54.8.4. Given 6 seemingly indistinguishable weights of 1, 2, 3, 4, 5 and 6 g, respectively, a drunken workman painted them at random “1 g”, . . . , “6 g”. How can you check whether the labels match the weights using only two weighings on a balance without any other weights except the given ones?

54.8.5. An air line was established between two countries so that any two cities, one from each country, are connected by precisely one flight which is a one-way flight and one can fly somewhere from each city. Prove that there are cities A, B, C, D , which can be visited by flying directly from A to B , from B to C , from C to D and from D to A .

Grade 9

54.9.1. Solve the equation:

$$(1 + x + x^2)(1 + x + \dots + x^{10}) = (1 + x + \dots + x^6)^2.$$

54.9.2. A conjurer divided a deck of a) 36, b) 54 cards into several piles and wrote a number equal to the number of cards in the pile on each card from every pile. Then he mixed the cards in a special way, divided them into piles once again and wrote another number equal to the number of cards in the new pile on each card to the right of the first number. Could the conjurer do this so that there are no equal pairs among the pairs of numbers on the cards and for every pair a, b there is a “symmetric” pair b, a ? (A pair a, a is assumed to be symmetric to itself.) Cf. Problem 54.10.5.

54.9.3. Prove that in a regular 12-gon $A_1A_2 \dots A_{12}$ the diagonals A_1A_5, A_2A_6, A_3A_8 and A_4A_{11} meet at one point.

54.9.4. After the graph of the function $y = \frac{1}{x}$ for positive x was drawn the coordinate axes were erased and their directions forgotten. How to recover the erased axes using a ruler and compass?

54.9.5. Cells of a 15×15 table contain nonzero numbers such that each of them is equal to the product of all neighboring numbers. (Two numbers are said to be neighboring if their cells have a common side.) Prove that all numbers in the table are positive.

Grade 10

54.10.1. A function f satisfies $f(x) + \left(x + \frac{1}{2}\right)f(1 - x) = 1$ for any $x \in (-\infty, \infty)$. Find a) $f(0)$ and $f(1)$; b) all such functions f .

54.10.2. What is the number n of identical billiard balls that can be arranged in space so that each ball is tangent to exactly three other balls? List all possible values of n .

54.10.3. Two nonintersecting circles are inscribed in a given angle. An isosceles triangle ABC is placed between the circles so that its vertices are on the sides of the angle and the equal sides AB and AC are tangent to the corresponding circles. Prove that the sum of the radii of the circles is equal to the height of the triangle drawn from vertex A .

54.10.4. We constructed a cube of size $10 \times 10 \times 10$ of 500 black and 500 white small identical cubes so that the cubes adjacent to each other were of different colors. Several of small cubes were removed from the cube so that exactly 1 small cube was missing in each of 300 rows or columns of size $1 \times 1 \times 10$ parallel to an edge of the cube. Prove that the number of black cubes removed is divisible by 4.

54.10.5. A conjurer divided a deck of 54 cards into several piles. A spectator writes the number equal to the number of cards in the corresponding pile on each card. Then the conjurer mixes the cards in a special way, divides them into piles again and the spectator writes another number equal to the number of cards in the new pile to the right of the first number on each card. They repeat this process several times. What is the least number of deals required for the conjurer to make different cards have different sets of numbers (whatever their position on the cards)?

Grade 11

54.11.1. Between which digits of the number 199...991 with 1991-many nines, one should insert a) + (the summation sign) to get the least possible number; b) \times (the multiplication sign) to get the greatest possible number?

54.11.2. Fig. 97 shows an orthogonal projection of the Earth (which is supposed to be an ideal ball) and its equator, A and B being the common points of the projection of the equator and the circle — the projection of the Earth).

FIGURE 97. (Probl. 54.11.2)

How can the projection of the North pole be found with the help of a ruler and compass?

54.11.3. Prove that in a regular 54-gon there are 4 diagonals that do not pass through the center and meet at one point.

54.11.4. A Parliament of 2000 MPs decided to ratify the state budget of 200 expenditure items. Each MP prepared a draft budget with what (s)he thinks the maximum possible allocation for each item so that the total expenditure does not exceed a given ceiling, S . For each item, the Parliament approves the maximum expenditure approved by not less than k MPs. What is the least value of k to ensure that the approved total does not exceed S ?

54.11.5. On a rectangular screen of size $m \times n$ divided into unit cells more than $(m - 1) \cdot (n - 1)$ cells are lighted. If in a 2×2 square 3 cells are not lighted, then the fourth cell switches itself off after a while. Prove that at least one cell of the screen is lighted at all times.

Olympiad 55¹ (1992)

Grade 8

55.8.1 (Se). Prove that if $a + b + c + d > 0$, $a > c$, $b > d$, then $|a + b| > |c + d|$.

¹The authorship of all problems of this olympiad is indicated after the number of the problem by an abbreviation boldfaced: **A. Galochkin**, **S. Gashkov**, **B. Kukushkin**, **I. Sergeev**, **I. Sharygin**, **A. Skopenkov**, **A. Spivak**, **S. Tokarev**.

55.8.2 (To). Can it happen during a game of chess that on each of 30 diagonals of the chess-board there stands an odd number of chips (each own number for each diagonal; some of these numbers may be equal)?

55.8.3 (To). An Olympiad lasted two days. Each participant solved during the first day as many problems as all other participants together during the second day. Prove that all participants of the Olympiad solved equal number of problems.

55.8.4 (To). What is the least number of weights in a set which can be divided into either 3, or 4, or 5 piles of equal mass?

55.8.5 (SG). Prove that in a right triangle the length of the bisector of the right angle does not exceed a half of the projection of the hypotenuse to the line perpendicular to the bisector.

55.8.6 (To). Are there four arrangements of 9 people at a round table such that no two of these people sit beside each other more than once? (Cf. Problem 55.9.6.)

Grade 9

55.9.1 (To). Each participant of a chess tournament won, as white, as many games as all remaining players together when they played as black. Prove that all participants won the same number of games each. (Cf. Problem 55.8.3.)

55.9.2 (AG). Which odd positive integers $n < 10000$ are more numerous: those for which the number formed by the four last digits of n^9 is greater than n or those for which it is smaller than n ?

55.9.3 (To). At the center of a square pie sits a raisin (of point size). A triangular piece can be cut off the pie along the line which intersects two neighboring sides of the square at the points different from vertices of the square; another triangular piece can be cut off the remaining part in the same manner, etc. Is it possible to cut the raisin off, i.e., to get a piece of the pie with the raisin? (Cf. Problem 55.10.2.)

55.9.4 (Sp). In a 9×9 square table, 9 cells are marked: those at the intersection of the 2-nd, 5-th and 8-th rows with the 2-nd, 5-th and 8-th column. In how many ways can one get from the lower left cell to the upper right one moving only upwards and to the right without entering marked cells?

55.9.5 (Sh). Diagonal AC of trapezoid $ABCD$ is equal to the lateral side CD . The line symmetric to BD with respect to AD intersects AC at point E . Prove that line AB divides DE in halves.

55.9.6 (To). Is it possible to place $2n + 1$ people at a round table n times so that no two sit beside each other more than once if (a) $n = 5$, (b) $n = 10$? (Cf. Problem 55.8.6.)

Grade 10

55.10.1 (AG). Prove that if the sum of cosines of the angles of a quadrilateral is equal to 0 then it is either a parallelogram, or a trapezoid, or an inscribed quadrilateral.

55.10.2 (To). A triangular piece can be cut off a pie of the form of a convex pentagon along the line that meets two neighboring sides at points distinct from the vertices; another piece can be cut off the remaining part in the same way, etc.

What are points on the surface of the pie should one stick a candle into so that it were impossible to get a piece of pie with the candle? (Cf. Problem 55.9.3.)

55.10.3 (AG). A white chip is placed in the bottom left corner of an $m \times n$ rectangular board, a black one is placed in the top right corner. Two players move their chips in turn along the horizontals or verticals 1 cell per move; the white can only move to the right or upwards. The white begins. The winner is the one who places his (her) chip on the cell occupied by the other player. Who can ensure the success: the white or the black?

55.10.4 (To). What is the least number of weights in the set that can be divided into either 4, 5, or 6 piles of equal mass? (Cf. Problem 55.8.4.)

55.10.5 (SG). Consider a convex centrally symmetric polygon. Prove that a rhombus of half the polygon's area can be placed inside the polygon.

55.10.6 (To). Each face of a convex polyhedron is a polygon with an even number of sides. Is it always possible to paint the edges of the polyhedron 2 colors so that each face has equal numbers of differently colored edges?

Grade 11

55.11.1 (To). It is required to place numbers into each cell of a $n \times n$ square table so that the sum of the numbers on each of $4n - 2$ diagonals were equal to 1. Is it possible to do this for (a) $n = 55$; (b) $n = 1992$?

55.11.2 (Ku). Find the angles of a convex quadrilateral $ABCD$ in which $\angle BAC = 30^\circ$, $\angle ACD = 40^\circ$, $\angle ADB = 50^\circ$, $\angle CBD = 60^\circ$ and $\angle ABC + \angle ADC > 180^\circ$.

55.11.3 (Sk). Aladdin visited every point of equator moving sometimes to the west, sometimes to the east and sometimes being instantaneously transported by genies to the diametrically opposite point on the Earth. Prove that there was a period of time during which the difference of distances moved by Aladdin to the west and to the east was not less than half the length of equator.

55.11.4 (Sp). Inside a tetrahedron a triangle is placed whose projections to the faces of tetrahedron are of area P_1, P_2, P_3, P_4 , respectively. Prove that

a) in a regular tetrahedron $P_1 \leq P_2 + P_3 + P_4$;

b) if S_1, S_2, S_3, S_4 are the areas of the corresponding faces of tetrahedron, then $P_1 S_1 \leq P_2 S_2 + P_3 S_3 + P_4 S_4$.

55.11.5 (To). Is it always possible to paint the edges of a convex polyhedron two colors so that for each face the number of edges painted one color would differ from the number of edges painted the other color by not more than 1?

55.11.6 (Se). A calculator can compare $\log_a b$ and $\log_c d$, where $a, b, c, d > 1$. It works according to the following rules:

if $b > a$ and $d > c$ the calculator passes to comparing $\log_a \frac{b}{a}$ with $\log_c \frac{d}{c}$;

if $b < a$ and $d < c$ the calculator passes to comparing $\log_c d$ with $\log_b a$;

if $(b - a)(d - c) \leq 0$ it prints the answer.

a) Show how the calculator compares $\log_{25} 75$ with $\log_{65} 260$. b) Prove that the calculator can compare two nonequal logarithms after finitely many steps.

Olympiad 56¹ (1993)

Grade 8

56.8.1 (Ku). Denote by $s(x)$ the sum of the digits of a positive integer x . Solve:

a) $x + s(x) + s(s(x)) = 1993$

b) $x + s(x) + s(s(x)) + s(s(s(x))) = 1993$

56.8.2 (Bo). Knowing that n is the sum of squares of three positive integers, prove that n^2 is also the sum of squares of three positive integers.

56.8.3 (Sl). On a straight line stand two chips, a red to the left of a blue. The Rule allows the following two operations: (a) to insert two chips of one color in a row at any place on the line and (b) to delete any two neighboring chips of one color. Is it possible to leave after finitely many operations only two chips on the line: a red to the right of a blue?

56.8.4 (Be). At the court of Tsar Gorokh, the tsar's astrologist declares a moment of time *favorable* if on a watch with a centrally placed second hand the minute hand occurs after the hour hand and before the second one (counting clockwise). Does the whole day (24 h) contain more favorable time than unfavorable?

56.8.5 (Sp). Is there a finite word composed of the letters of Russian alphabet (32 letters) that has no two identical neighboring subwords but such subwords appear if one ascribes any letter (of the same alphabet) in front or at the back of this word?

56.8.6 (Ak). A circle centered at D passes through points A, B , and the center O of the escribed circle of triangle $\triangle ABC$ tangent to side BC and the extensions of sides AB and AC . Prove that points A, B, C , and D lie on one circle.

Grade 9

56.9.1 (Sh). For distinct points A and B on a plane, find the locus of points C such that triangle $\triangle ABC$ is acute and the value of its angle $\angle A$ is intermediate among the triangle's angles.

56.9.2 (Ko). Let $x_1 = 4$, $x_2 = 6$ and define x_n for $n \geq 3$ to be the least non-prime greater than $2x_{n-1} - x_{n-2}$. Find x_{1000} .

¹The authorship of all problems of this olympiad is indicated after the number of the problem by an abbreviation boldfaced: I. F. **Akulich**, A. **Andzhans**, A. Ya. **Belov**, A. I. **Galochkin**, **G. Galperin**, S. B. **Gashkov**, **GZ-B**: S. M. Gusein-Zade, A. Ya. **Belov**, G. **Kondakov**, S. V. **Konyagin**, B. N. **Kukushkin**, D. **Botin**, **Slitinsky**, A. W. **Spivak**, I. F. **Sharygin**, S. I. **Tokarev**, **VI**: A. Vladimirov, R. Ismailov, **VT**: M. Vyaly, D. Tereshin.

56.9.3 (Gal). A paper triangle with angles of 20° , 20° , 140° is cut along one of its bisectors into two triangles; one of these triangles is also cut along one of its bisectors, etc. Can we obtain a triangle similar to the initial one after several cuts?

56.9.4 (To). In Pete's class there are 28 students beside him. Each two of these 28 have a distinct number of friends among the classmates. How many friends does Pete have in this class?

56.9.5 (GG). To every pair of numbers x, y the Rule assigns a number $x * y$. Find $1993 * 1935$ if it is known that

$$x * x = 0, \text{ and } x * (y * z) = (x * y) + z \text{ for any } x, y, z.$$

56.9.6 (Sh). Given a convex quadrilateral $ABMC$ with $AB = BC$, $\angle BAM = 30^\circ$, $\angle ACM = 150^\circ$, prove that AM is the bisector of $\angle BMC$.

Grade 10

56.10.1 (Ga). In the representation of numbers A and B as decimal fractions the lengths of their minimal periods are equal to 6 and 12, respectively. What might the length of the minimal period in the similar representation of $A + B$ be? Find all answers.

56.10.2 (Ga). The grandfather of Baron K. F. I. von Münchhausen constructed a castle with a square in the horizontal cross-section. He divided the castle into 9 equal square ball rooms and placed the arsenal in the middle one. Baron's father divided each of the remaining 8 ball rooms into 9 equal square halls and organized winter gardens in all central halls. Baron himself divided each of the 64 empty halls into 9 equal square rooms and placed a swimming pool in each of the central rooms. Baron furnished the other rooms and made a door between every pair of neighboring furnished rooms. Baron shut all the other temporary doors.

Baron boasts that he once managed to go over his furnished rooms visiting each just once and returning in the initial one. We know Baron as a gentleman with a name for honesty won by his truthful stories, but still wonder: is he telling the truth in this instance?

56.10.3 (Kon). A river connects two circular lakes of radius 10 km each; the banks of the river and the lakes are segments of either straight lines or circles. From any point on any of the river's banks one can take a boat and reach the other bank by swimming not longer than 1 km. Assuming that the boat is a point is it possible for a pilot to lead the boat along the river in order to be at the distance of not more than (a) 700 m (b) 800 m away from each of the banks?

56.10.4 (VI). For every pair of real numbers a and b consider the sequence¹ $p_n = [2\{an + b\}]$. Any k successive terms of this sequence is called a *word*. Is it true that any ordered set of 0's and 1's of length k can be a word of the sequence determined by certain a and b for (a) $k = 4$, (b) $k = 5$?

56.10.5 (VT). In a botanical classifier a plant is determined by 100 features. Each of the features can either be present or absent. A classifier is considered to be *good* if any two plants have less than half of the features in common. Prove that a good classifier can not describe more than 50 plants.

56.10.6 (Sh). On side AB of triangle ABC the square is constructed outwards, its center is O . Points M and N are the midpoints of AC and BC ; the lengths of these sides are equal to a and b , respectively. Find the maximum of the sum $OM + ON$ as the angle $\angle ACB$ varies.

Grade 11

56.11.1 (Be). Knowing that $\tan \alpha + \tan \beta = p$ and $\cot \alpha + \cot \beta = q$ find $\tan(\alpha + \beta)$.

56.11.2 (Be). The unit square is divided into finitely many smaller squares (of, perhaps, distinct sizes). Consider the squares whose intersection with the main diagonal is nonempty. Is it possible for the sum of perimeters of the squares to be greater than 1993?

56.11.3 (An). Given n points on a plane no three of which lie on one line. A straight line passes through every pair of the points. What is the least number of pair-wise non-parallel lines among these lines?

56.11.4 (GZB). Stones lie in several boxes. The Rule allows us in one move: to select a number n ; to unite the stones in each box in groups of n and a residue of less than n stones in it; to leave in each group a stone and the whole residue; it also allows us to pocket the rest of the stones. Is it possible to ensure in 5 moves that each box contains one stone if initially there were not more than (a) 460 stones, (b) 461 stones in each box?

¹Recall that $\{x\}$ and $[x]$ denotes the fractional and the integer part of x , respectively.

56.11.5 (Be). It is known that the domain of definition of a function f is segment $[-1, 1]$, and $f(f(x)) = -x$ for all x ; the graph of f is the union of finitely many points and intervals.

Is it possible to draw the graph of f if the domain of f is a) $]-1, 1[$? b) the whole real line?

56.11.6 (Sh). A fly flies inside a regular tetrahedron with edge a . What is the shortest length of the flight the fly should take to visit every face and return to the initial spot?

Olympiad 57¹ (1994)

Grade 6

57.6.1. Can there be four people among which no three have identical first name, patronimic (middle name) and the last name but any pair of these people has identical either first, or middle, or last name?

57.6.2. Find a) the 6-th, b) the 1994-th number in the sequence 2, 6, 12, 20, 30, ...

57.6.3. Several teams of guards of social property, manned by identical number of guards each, slept more nights during their vigil than there are guards in the team but less than there are teams. How many guards are there in the team if all guards from the team together slept 1001 man-night?

57.6.4. Construct a $3 \times 3 \times 3$ cube of $1 \times 1 \times 1$ red, green and yellow cubes so that in any $3 \times 1 \times 1$ layer there are cubes of all three colors.

57.6.5. Cut a square into three parts from which it is possible to construct a nonright scalene triangle.

57.6.6. Kate's family drank coffee. Each member of the family drank out a full cup of coffee with milk and Kate drank a quarter of the milk and a sixth of the coffee. How many people are there in Kate's family?

57.6.7. Among any 9 of 60 kids three are from the same grade. Is it necessary that there are a) 15, b) 16 kids from the same grade?

57.6.8. A pedestrian walked along (across?) 6 streets of a town in a row passing each street exactly twice; however long he contemplated over the map he could not find a route so as to pass along any street just once during one stroll. Is there such a route?

Grade 7

57.7.1. During the past two years a factory lowered the volume of the products it manufactured by 51%. Each year the volume diminished by the same number of percents. What is this number? (5 points)

57.7.2. Each staircase of a house has the same number of floors; the same number of apartments on each floor. There are more floors than the number of apartments on the floor; more apartments on the floor than there are staircases and there is more than one staircase. How many floors are there in the house if the total number of its apartments is 105?

a) Find at least one solution. (2 points)

b) Find all solutions and prove that there are no more. (4 points)

57.7.3. When the committee asked Neznajka (Master Ignoramus) to contribute with a problem for a Math Olympiad in the Sunny Town he wrote the following head-twister, where different letters replace different figures:

$$\begin{array}{r} + \quad ABC \\ \quad DEF \\ \hline \quad GHL \end{array}$$

Is it possible to solve it? (5 points)

57.7.4. There are plenty of red, green and yellow cubes of size $1 \times 1 \times 1$. Is it possible to compose of them a $3 \times 3 \times 3$ cube so that each $3 \times 1 \times 1$ layer has all three colors? (6 points)

57.7.5. On a 4×6 board there stand two Ivan's black chips and two Sergey's white chips (as on Fig. 98. a)).

Each player, in turn, moves any of his chips one step along the vertical. Ivan, though plays black, starts. If after somebody's move a black chip occurs among two white ones along either a horizontal or a diagonal (as on Fig. 98. b)) it is considered *killed* and should be removed from the board. Ivan's goal is to lead his chips from the top row to the bottom one. Can Sergey prevent Ivan from getting his goal? (8 points)

¹Problems for the 6-th and 7-th grades were selected by a committee headed by D. Bochina, S. Dorichenko, A. Kovaldzhii and I. Yashchenko; the authors of the other problems are: A. Kovaldzhii (8.2, 8.4, 10.1), D. Botin (8.5), Yu. Chekanov (9.2), G. Galperin (11.4), A. Galochkin (9.5), K. Ignatiev (9.6), A. Kovaldzhii, G. Kondakov (11.1), O. Kryzhanovski (8.6, 10.6), S. Markelov (11.5), I. Nagel (9.4), V. Proizvolov (10.5), G. Shabat (10.2), I. Sharygin (10.4), N. Vasiliev (9.3, 11.3, 11.6); the authors of several problems are anonymous.

FIGURE 98. (Probl. 57.7.5)

57.7.6. In a school the astronomical circle gathered 20 times. Each time there were exactly 5 listeners and no 2 students met during the circle's gatherings more than once. Prove that at least 20 different students attended the circle. (12 points)

Grade 8

57.8.1. A cooperative enterprize gets apple and grape juice in identical cans and produces a mixed drink in equal jars. One can of apple juice suffices for exactly 6 jars of the drink; one can of grape juice suffices for exactly 10 jars of the drink. When they changed the recipe one can of apple juice became sufficient for exactly 5 jars of the drink only. For how many jars of the drink will now suffice one can of the grape juice? (The drink is a pure mixture not diluted with water or preservatives, etc.)

57.8.2. A student did not notice a multiplication sign between two three-digit numbers and wrote one 6-digit number that happened to be 7 times greater than the product of the two three-digit numbers. Find the factors.

57.8.3. In a triangle ABC the bisectors of angles A and C are drawn. Points P and Q are the bases of the perpendiculars dropped from vertex B to these bisectors. Prove that $PQ \parallel AC$.

57.8.4. Four grasshoppers sit at the vertices of a square. Each minute one of them hops into the point symmetric with respect to another grasshopper. Prove that it is impossible for the grasshoppers to sit at some moment at the vertices of a larger square.

57.8.5. The royal astrologer considers a moment of time *favorable* if the hour, minute and second hands of the clock are on one side of the dial's diameter. All other time is considered *unfavorable*. The hands turn around a common axis uniformly, without jumps. Which kind of time prevails during the full day (24 hours), favorable or unfavorable?

57.8.6. Two play a game on a 19×94 checkered board. Each in turn marks a square (of any possible size) along the lines of the mesh and shades it. The one who shades the last cell wins. It is forbidden to shade a cell twice. Who wins if played optimally and what should the strategy be?

Grade 9

57.9.1. Is there a nonconvex pentagon no two of whose five diagonals have a common point apart from a vertex?

57.9.2. Kolya has a line segment of length k , Leo has another one, of length l . First, Kolya divides his segment into three parts; then Leo divides his segment into three parts. If it is possible to build two triangles from the six segments obtained, Leo wins; otherwise Kolya wins. Depending on the ratio $\frac{k}{l}$, who, Kolya or Leo, can assure victory and what should the winning strategy be?

57.9.3. Prove that the equation

$$x^2 + y^2 + z^2 = x^3 + y^3 + z^3$$

has infinitely many solutions in integers.

57.9.4. Two circles intersect at points A and B . To both circles tangents are drawn through A . The tangents intersect the circles at points M and N . The straight lines intersect the circles again at points P and Q (P lies on BM , Q lies on BN). Prove that $MP = NQ$.

57.9.5. Find the maximal natural number not ending with a 0 such that if we strike out one (not the first) of its figures we get a divisor of the initial number.

57.9.6. During dull lessons students sometimes play “marine battle”. In a 10×10 square of checkered paper one should place ships — rectangles — of sizes: one 1×4 , two 1×3 , three 1×2 and four 1×1 . The ships should not have common points (even vertices) but can have common points (even edges) with the sides of the square. Prove that

a) if one places the ships as listed above (starting with the largest), one can always squeeze all the ships in the square even if one lives the running moment at all times and places each ship without thinking about the other ships’ future;

b) if one places the ships in the opposit order (starting with smaller ships), a situation might arise when it is impossible to squeeze in the next ship. (Give an example.)

Grade 10

57.10.1. A student did not notice the multiplication sign between two 7-digit numbers and wrote one 14-digit number which turned out to be 3 times the would be product. What are the initial numbers?

57.10.2. An infinite sequence of numbers x_n is determined by the formula

$$x_{n+1} = 1 - |1 - 2x_n|, \quad 0 \leq x_1 \leq 1.$$

Prove that the sequence is periodic starting from a certain place a) if and b) only if $x_1 \in \mathbb{Q}$.

57.10.3. Each of the 1994 Parliament members slapped exactly one of his/her colleagues on the face. Prove that it is possible to compose a Parliament Committee of 665 members none of whom settles disputes with the colleagues in this way.

57.10.4. Let D be a point on side BC of $\triangle ABC$. Circles are drawn inside $\triangle ABD$ and $\triangle ACD$; a common outer tangent (distinct from BC) is drawn to the circles; it intersects AD at K . Prove that the length of AK does not depend on the position of D on BC .

57.10.5. Consider an arbitrary polygon, not even necessarily convex one. Recall that a *chord* of a polygon is a line segment whose endpoints belong to the polygon’s contour while the segment itself lies entirely inside the polygon, the contour included.

a) Is there always a chord of the polygon that divides it into parts of equal area?

b) Prove that any polygon can be divided by a chord into parts the area of each of them not less than $\frac{1}{3}$ of the total area of the polygon. (We always assume that a chord divides the polygon into two parts: the part that splits into several pieces is, nevertheless, considered as one part.)

57.10.6. Is there a polynomial $P(x)$ with a negative coefficient while all the coefficients of any power $P^n(x)$ are positive for $n > 1$?

Grade 11

57.11.1. Devise a polyhedron with no three faces having the same number of edges.

57.11.2. See Probl. 57.10.2

57.11.3. In a round goblet whose section is the graph of the function $y = x^4$ a cherry — a ball of radius r — is dropped. What is the largest r for which the ball can touch the lowest point of the bottom? (In plain math words: what is the maximal radius of the disc lying in the domain $y \geq x^4$ and containing the origin?)

57.11.4. A convex polyhedron has 9 vertices, one of which is A . Parallel translations that send A into each of the other vertices form 8 equal polyhedra. Prove that at least two of these 8 polyhedra have an inner point in intersection.

57.11.5. Extensions of the sides AB and CD of a convex polygon $ABCD$ intersect at point P ; extensions of the sides BC and AD intersect at point Q . Prove that if each of the Consider three pairs of bisectors: the outer angles of the quadrilateral at vertices A and C ; the outer angles at vertices B and D ; and the outer angles at vertices P and Q of triangles $\triangle QAB$ and $\triangle PBC$, respectively. Prove that if each of the three pairs of bisectors intersects, the intersection points lie on one straight line.

57.11.6. Prove that for any $k > 1$ there exists a power of 2 such that among its k last digits the nines constitute not less than one half. For example: $2^{12} = 4096$, $2^{53} = \dots 992$

Olympiad 58¹ (1995)

Grade 8

58.8.1. M. V. Lomonosov spent one denezhka a day for a loaf of bread and kvas. When prices went up 20%, he bought half a loaf of bread and kvas for the same denezhka. Will a denezhka be enough to buy at least kvas if the prices will again rise 20%?

58.8.2. Prove that the numbers of the form 10017, 100117, 1001117,... are divisible by 53.

58.8.3. Consider a convex quadrilateral and a point O inside it such that $\angle AOB = \angle COD = 120^\circ$, $AO = OB$ and $CO = OD$. Let K , L and M be the midpoints of sides AB , BC and CD , respectively. Prove that a) $KL = LM$; b) triangle KLM is an equilateral one.

58.8.4. To manufacture a parallelepipedal closed box of volume at least 1995 units we have a) 962, b) 960, c) 958 square units of material. Assuming our production is wasteless, is the stock sufficient?

58.8.5. Several villages are connected with a town; there is no direct communication between villages. A truck with goods for all villages starts from the town. The cost of the truck's trip is equal to the product of the total weight of the load by the distance. Suppose that the weight of each item in the load is equal in some units to the distance from the town to the item's destination. Prove that the cost of the delivery does not depend on the order in which the goods are delivered.

58.8.6. A straight line cuts off a regular quadrilateral $AB C D E F$ triangle $A K N$ such that $A K + A N = A B$. Find the sum of the angles with vertices in the vertices of the quadrilateral that subtend segment $K N$.

Grade 9

58.9.1. Prove that if we insert any number of digits 3 between the zeroes of the number 12008, we get a number divisible by 19.

58.9.2. Consider an isosceles triangle $A B C$. For an arbitrary point P inside the triangle consider intersection points A' and C' of straight lines $A P$ with $B C$ and $C P$ with $B A$, respectively. Find the locus of points P for which segments $A A'$ and $C C'$ are equal.

58.9.3. Let us refer to a rectangular of size $1 \times k$ for any natural k a *strip*. For what integer n can one cut a $1995 \times n$ rectangle into pairwise different strips?

58.9.4. Consider a quadruple of natural numbers a , b , c and d such that $ab = cd$. Can $a + b + c + d$ be a prime?

58.9.5. We start with four identical right triangles. In one move we can cut one of the triangles along the height from the right angle into two triangles; so we get 5 right triangles. Prove that after any number of moves there are two identical triangles among the whole lot.

58.9.6. Geologists took 80 cans with preserved food for a trip. The weights of cans are known and pairwise distinct (there is an inventory). After a while the labells became unreadable and only the cook knows which can contains what. She can prove it beyond any doubt without opening the cans and using only the list of inventory and a balance with two pans and a hand that shows the difference of weight in the pans. Prove that to this end a) 4 weighings suffice while b) 3 do not.

58.9.1. The number $\sin a$ is known. What is the largest number of different values that a) $\sin \frac{a}{2}$? b) $\sin \frac{a}{3}$ can take?

Grade 10

58.10.2. See Probl. 58.9.2.

58.10.3. Consider trapezoid $A B C D$. We construct circles with the lateral sides of the trapezoid as diameters. Suppose that the diagonals of $A B C D$ meet at point K not on these circles. Prove that the lengths of the tangents to these circles from point K are equal.

58.10.4. See Probl. 58.9.5.

58.10.5. Prove that if a , b and c are integers and, moreover, $\frac{a}{b} + \frac{b}{c} + \frac{c}{a}$ and $\frac{a}{c} + \frac{c}{b} + \frac{b}{a}$ are integers, then $a = b = c$.

¹The authors of the problems are: A. Belov (10.6, 11.5, 11.7), D. Botin(8.4), Yu. Chekanov (9.3), A. Galochkin (8.2, 9.1, 11.1, 11.2), A. Gribalko (10.5), G. Kondakov (11.4), W. K. Kovaldzhi (8.1, 8.5), S. Markelov (8.3, 9.2, 10.1, 10.3), A. Shapovalov (9.5), V. Senderov (10.6, 11.5, 11.7), I. Sharygin (11.3), V. Proizvolov (8.6), A. Tolpygo (9.6).

58.10.6. On a board, several bulbs are on. There are several buttons on the control panell. Pressing a button changes the state of the bulbs it is connected with. It is known that for any collection of bulbs there is a bulb connected with an odd number of bulbs from this set. Prove that by pressing buttons on can switch off all the bulbs.

Grade 11

58.11.1. Prove that $|x + y + z| \leq |x + y - z| + |x - y + z| + |-x + y + z|$, where x, y, z are real numbers.

58.11.2. Is it possible to paint the edges of n -angled prizm 3 colors so that each face had the boundary painted all 3 colors and each vertex was the intersection point of edges of different colors if a) $n = 1995$, b) $n = 1996$?

58.11.3. Consider triangle ABC , its median AM , bisector AL and a point K on AM such that $KL \parallel AC$. Prove that $AL \perp KC$.

58.11.4. Divide segment $[-1, 1]$ into black and white subsegments so that the integral of any a) linear function, b) quadratic polynomial along black segments was equal to that along white ones.

58.11.5. Consider two infinite in both ways sequences A of period 1995 and B which is either nonperiodic or the length of its period is $\neq 1995$. Let any segment of sequence B not longer than n be contained in A . What is the largest n for which such sequences exist?

58.11.6. Prove that there exist infinitely many nonprime n 's such that $3^{n-1} - 2^{n-1} \vdots n$.

58.11.7. Is there a polygon and a point outside it such that from this point non of its vertices is visible?

Olympiad 59¹ (1996)

Grade 8

59.8.1. It is known that $a + b^2/a = b + a^2/b$. Is it true that $a = b$? (*R Fedorov*)

59.8.2. Along a circle stand 10 iron weighs. Between every two weighs there is a brass ball. Mass of each ball is equal to the difference of masses of its neighboring weighs. Prove that it is possible to divide the balls among two pans, so as to make the balance in equilibrium. (*V. Proizvolov*)

59.8.3. At nodes of graph paper gardeners live; flowers grow everywhere around them. Each flower is to be taken care of by the three nearest to it gardeners. One of the gardeners wishes to know what is the flower (s)he has to take care of. Sketch the plot of these gardeners. (*I. F. Sharygin*)

59.8.4. Consider an equilateral triangle $\triangle ABC$. The points K and L divide the leg BC into three equal parts, the point M divides the leg AC in ratio $1 : 2$ counting from the vertex A . Prove that $\angle AKM + \angle ALM = 30^\circ$. (*V. Proizvolov*)

59.8.5. A rook stands in a corner of an $n \times n$ chess board. For what n , moving alternately along horizontals and verticals, the rook can visit all the cells of the board and return to the initial corner after n^2 moves? (A cell is *visited* only if the rook stops on it, those that the rook "flew over" during the move are not counted as visited.) (*A. Spivak*)

59.8.6. Eight students solved 8 problems. a) It turned out that each problem was solved by 5 students. Prove that there are two students such that each problem is solved by at least one of them.

b) If it turned out that each problem was solved by 4 students, it can happen that there is no pair of students such that each problem is solved by at least one of them. (Give an example.) (*S. Tokarev*)

Grade 9

59.9.1. Numbers a, b and c satisfy inequalities $|a - b| \geq |c|$, $|b - c| \geq |a|$, $|c - a| \geq |b|$. Prove that one of the numbers a, b or c is equal to the sum of the other two numbers. (*A. Galochkin*)

59.9.2. The circle is circumscribed about $\triangle ABC$; through the points A and B tangents are drawn, they meet at M . The point N lies on the leg BC , and $MN \parallel AC$. Prove that $AN = NC$. (*I. F. Sharygin*)

59.9.3. Integers 1 to n are written in a row. Under them, the same numbers are written in some other order. Could it happen that the sum of each number with the one under it is a perfect square? Consider a) $n = 9$, b) $n = 11$, c) $n = 1996$? (*P. Filevich*)

59.9.4. Let A and B be points on a circle. They divide the circle into two parts. Find the locus of the midpoints of all chords whose endpoints lie on different arcs $\cup AB$. (*I. F. Sharygin*)

59.9.5. Ali-Baba and a robber divide a treasure consisting of 100 golden coins. The treasure is split into 10 piles of 10 coins. Ali-Baba chooses 4 piles, places a mug beside each pile, and puts several coins (not less than 1, but not the whole pile) from the respective pile into each mug. The robber must rearrange the mugs by altering their initial attribution to piles, after which the coins are taken out from each mug and added to the newly attributed pile.

Then Ali-Baba again selects 4 piles of 10, places mugs beside the piles, etc.

At any moment Ali-Baba can quit and go away with any 3 mugs he chooses. The remaining coins will be the robber's share. What is the greatest number of coins Ali-Baba can collect, if the robber is no altruist either? (*A. Ja. Belov*)

Grade 10

59.10.1. Positive numbers a , b and c satisfy $a^2 + b^2 - ab = c^2$. Prove that $(a - c)(b - c) \leq 0$. (*A. Egorov, V. Bugaenko*)

59.10.2. In a 10×10 square drawn on a graph paper along its lines, the centers of all unit squares (100 points altogether) are marked. What is the least number of straight lines non parallel to the sides of big square and passing through all the points marked? (*A. Shapovalov*)

59.10.3. The points P_1, P_2, \dots, P_{n-1} divide the side BC of an equilateral triangle $\triangle ABC$ into n equal segments: $BP_1 = P_1P_2 = \dots = P_{n-1}C$. The point M on the side AC is such that $AM = BP_1$. Prove that $\angle AP_1M + \angle AP_2M + \dots + \angle AP_{n-1}M = 30^\circ$, if a) $n = 3$; b) n is an arbitrary positive integer. (*V. Proizvolov*)

59.10.4. In a corner of an $m \times n$ chessboard stands a bishop. Two play in turns; they alternately move the bishop horizontally or vertically any distance; the Rule forbids the bishop to stop on the field over which it had been already moved or at which it had already stopped. The one who is stuck is the loser. Which player can assure victory for him/herself: the one who starts or the other one and now should (s)he move? (*B. Begun*)

59.10.5. In a country, the houses of the inhabitants being represented by points on the plane, two Laws act:

- 1) A person can play basketbol only if (s)he is taller the majority of his/her neighbors.
- 2) A person has the right for free usage of the public transport only if (s)he is shorter the majority of his/her neighbors.

According to Law, the person's *neighbors* are the inhabitants living in side the circle centered at the person's house. The humane Law lets each person to chose the radii for each section of the Law. Can not less than 90% of the population play basketbol and not less than 90% have the right for free usage of the public transport? (*N. N. Konstantinov*)

59.10.6. Prove that for any n th degree polynomial $P(x)$ with natural coefficients there exists a k such that the numbers $P(k), P(k+1), \dots, P(k+1996)$ are not prime ones, if a) $n = 1$; b) n is an arbitray positive integer. (*V. A. Senderov*)

Grade 11

59.11.1. Positive numbers a , b and c satisfy equation $a^2 + b^2 - ab = c^2$. Prove that $(a - c)(b - c) \leq 0$. (*A. Egorov, V. Bugaenko*)

59.11.2. Find a polynomial with integer coefficients whose roots are $\sqrt[5]{2 + \sqrt{3}} + \sqrt[5]{2 - \sqrt{3}}$. (*B. Kukushkin*)

59.11.3. In space, consider 8 parallel planes such that??? the distances between each two neighboring ones are equal. A point is selected on each of the planes. Can the points selected be vertices of a cube? (*V. Proizvolov*)

59.11.4. Prove that there are infinitely many natural numbers n such that n is representable as the sum of squares of two natuatural numbers, while $n - 1$ and $n + 1$ are not. (*V. A. Senderov*)

59.11.5. Point X outside of nonintersecting circles, ω_1 and ω_2 , is such that the segments of the tangents drawn from X to ω_1 and ω_2 are equal. Prove that the intersection point of the diagonals of the quadrilateral, determined by the tangent points, coincides with the intersection point of the common inner tangents to ω_1 and ω_2 . (*S. Markelov*)

59.11.6. A $2^n \times n$ table consists of all possible lines of length n composed from numbers 1 and -1 . Part of the numbers was replaced with zeros. Prove that one can choose several lines whose sum (if we consider each line as a number) is zero. (*G. Kondakov*)

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Grade 8

60.8.1. In certain cells of the chess board stand some figures. It is known that on each horizontal line stands at least one figure and on different horizontals a different number of figures stand. Prove that it is possible to mark 8 figures so that on each horizontal and each vertical stands exactly one marked figure. (*V. Proizvolov*)

60.8.2. From a volcano observatory to the top of Stromboli volcano one has to take a road and then a passway, each takes 4 hours. There are two craters on the top. The first crater erupts for 1 hour and then is silent for 17 hours, next all over again, it erupts for 1 hour and then is silent for 17 hours, etc. The second crater erupts for 1 hour and then is silent for 9 hours, then it erupts for 1 hour, then is silent for 17 hours, etc. During the eruption of the first crater it is dangerous to take both the passway and the road, during the eruption of the second crater it is dangerous to take the passway only. At noon scout Vanya saw that both the craters simultaneously started to erupt. Will it be ever possible for him to mount the top of the volcano without risking his life? (*I. Yashchenko*)

60.8.3. Inside of the acute angle $\angle XOY$ points M and N are taken so that $\angle XON = \angle YOM$. On the segment OX a point Q is taken so that $\angle NQO = \angle MQX$; on segment OY a point P is taken so that $\angle NPO = \angle MPY$. Prove that the lengths of the broken lines MPN and MQN are equal. (*V. Proizvolov*)

60.8.4. Prove that there exists a positive non-prime integer such that if any three of its neighboring digits are replaced with any given triple of digits the number remains non-prime. Does there exist a 1997-digit such number? (*A. Shapovalov*)

60.8.5. In the rhombus $ABCD$ the measure of $\angle B = 40^\circ$, E is the midpoint of BC , F is the base of the perpendicular dropped from A on DE . Find the measure of $\angle DFC$. (*M. Volchkevich*)

60.8.6. Banker learned that among similarly looking golden coins one is counterfeit (of less weight). Banker asked an expert to determine the coin by means of a balance without weights and demanded that each coin should participate in not more than two weighings (otherwise it will get too worn out and lose its market value). What largest number of coins should Banker have had to ensure the fulfilment of the expert's task? (*A. Shapovalov*)

Grade 9

60.9.1. In a triangle one side is 3 times shorter than the sum of the other two. Prove that the angle opposite the said side is the smallest of the triangle's angles. (*A. Tolpygo*)

60.9.2. On a plate lie 9 different pieces of cheese. Is it always possible to cut one of them into two parts so that the 10 pieces obtained were divisible into two portions of equal mass of 5 pieces each? (*V. Dolnikov*)

60.9.3. A convex octagon $AC_1BA_1CB_1$ satisfies: $AB_1 = AC_1$, $BC_1 = BA_1$, $CA_1 = CB_1$ and $\angle A + \angle B + \angle C = \angle A_1 + \angle B_1 + \angle C_1$. Prove that the area of $\triangle ABC$ is equal to a half area of the octagon. (*V. Proizvolov*)

60.9.4. Along a circular railroad n trains circulate in the same direction and at equal distances between them. Stations A , B and C on this railroad (denoted as the trains pass them) form an equilateral triangle. Ira enters station A at the same time as Alex enters station B in order to take the nearest train. It is known that if they enter the stations at the same moment of time as the driver Roma passes a forest, then Ira takes her train earlier than Alex; otherwise Alex takes the train earlier than or simultaneously with Ira. What part of the railroad goes through the forest? (*V. Proizvolov*)

60.9.5. $2n$ sportsmen twice met at a circle tournament. Prove that if the sum of points of each altered not less by n (during the second tournament), it altered by exactly n . (*V. Proizvolov*)

60.9.5. Let $1 + x + x^2 + \dots + x^{n-1} = F(x)G(x)$, where $n > 1$ and where F and G are polynomials, whose coefficients are zeroes and units. Prove that one of the polynomials F and G can be represented in the form $(1 + x + x^2 + \dots + x^{k-1})T(x)$, where $k > 1$ and where T is also a polynomial whose coefficients are zeroes and units. (*V. Senderov, M. Vyaly*)

Grade 1

0

60.1.1. Is there a convex body distinct from ball whose three orthogonal projections on three pairwise perpendicular planes are discs? (*A. Kanel-Belov*)

60.1.2. Prove that among the quadrilaterals with given lengths of the diagonals and the angle between them the parallelogram has the least perimeter. (*Folklore*)

60.1.3. Consider a quadrilateral. a) As the quadrilateral was circumvented clockwise, each side of the quadrilateral was extended by its length in the direction of the movement. It turned out that the endpoints of the segments constructed are the vertices of a square. Prove that the initial quadrilateral is a square.

b) Prove that if as a result of the procedure similar to the above-described is applicable to an n -gon we get a regular n -gon, then the initial n -gon is a regular one. (*M. Evdokimov*)

60.1.4. Given real numbers $a_1 \leq a_2 \leq a_3$ and $b_1 \leq b_2 \leq b_3$ such that

$$\begin{aligned} a_1 + a_2 + a_3 &= b_1 + b_2 + b_3, \\ a_1 a_2 + a_2 a_3 + a_1 a_3 &= b_1 b_2 + b_2 b_3 + b_1 b_3. \end{aligned}$$

Prove that if $a_1 \leq b_1$ and $a_3 \leq b_3$. (*Folklore*)

60.1.5. In a circle tournament with more than two participants the *coefficient* of each participant was defined to be the sum of points scored by those defeated by the sportsman considered. It turned out that the coefficients of all participants are equal. Prove that all the participants scored equal number of points. (*B. Frenkin*)

60.1.6. Consider the powers of 5: 1, 5, 25, 125, 625, ... Consider the sequence formed by their first digits: 1, 5, 2, 1, 6, ... Prove that any segment of this sequence written in reverse order will be encountered in the sequence of the first digits of the powers of 2: 1, 2, 4, 8, 1, 3, 6, 1, ... (*A. Kanel-Belov*)

Grade 1

1

60.1.1. On sides AB , BC and CA of $\triangle ABC$ points C' , A' and B' , respectively, are marked. Prove that the area of $\triangle A'B'C'$ is equal to

$$\frac{AB' \cdot BC' \cdot CA' + AC' \cdot CB' \cdot BA'}{4R},$$

where R is the radius of the circumscribed circle of $\triangle ABC$. (*A. Zaslavsky*)

60.1.2. Compute

$$\int_0^{\pi/2} (\cos^2(\cos x) + \sin^2(\sin x)) dx.$$

where R is the radius of the circumscribed circle of $\triangle ABC$. (*M. Vyaly*)

60.1.3. Consider three functions:

$$f_1(x) = x + \frac{1}{x}, \quad f_2(x) = x^2, \quad f_3(x) = (x-1)^2.$$

The Rule lets you to add subtract and multiply these functions (in particular, you can square and raise to higher powers, etc.), multiply by an arbitrary number, add an arbitrary number to your result and perform the above described operations with the expressions obtained. Get in this way $\frac{1}{x}$. Prove that if one of the functions f_1 , f_2 or f_3 is taken out of the consideration, then it is impossible to get $\frac{1}{x}$ in the way described. (*M. Evdokimov*)

60.1.4. Is it possible to divide a regular tetrahedron with edge 1 into regular tetrahedrons and octahedrons with the lengths of their edges less than $\frac{1}{100}$? (*V. Proizvolov*)

60.1.5. Positive numbers a , b and c are such that $abc < 1$. Prove that

$$\frac{1}{1+a+b} + \frac{1}{1+b+c} + \frac{1}{1+c+a} \leq 1.$$

(*G. Galperin*)

60.1.6. On the plane, consider a finite number of strips with the sum of their widths equal to 100 and a disc of radius 1. Prove that it is possible to translate parallelly each strip so that the totality of translated strips will cover the disc. (*M. Smurov*)

Selected problems of Moscow mathematical circles

The following are problems we find most interesting among those offered to the participants of mathematical clubs, to the winners of the Moscow Olympiads when they were coached to International Olympiads and also some problems from the archives of the Moscow Olympiad jury which were not used in any of the tournaments, and, therefore, are not well known. The grade for which the problem was intended is given in parentheses.

1.(7-9). a) Find the sum of the digits of the number $123456789101112\dots 999998999999$.

b) How many digits 7 are there in this number?

2.(8-10). Find the least positive integer $\overline{ab\dots c}$ without any zeroes in its decimal notation, such that its sum with itself written in reverse order (i.e. the sum with the number $\overline{c\dots ba}$), is a number whose digits can be obtained by a permutation of the digits of the original number.

3.(9-10). Are there irrational numbers x and y for which x^y is a rational number?

4.(8-10). We place 1's and -1 's at the vertices of a cube, one number per vertex. In the center of each face we put the product of the numbers at the vertices of this face. Can the sum of the 14 numbers obtained be equal to 0? Can it be equal to 7?

5.(8-10). a) There is a finite number of stars in space. The number of and directions to visible stars can be determined from an observation post. No single observation, however, determines the exact number of stars, as some might be hidden behind the others. It is only possible to say, after several observations, that the number of stars in the sky is not less than the greatest of the numbers of stars visible from observation posts. Can the exact number of stars in the sky be still determined after several observations? If so, what is the least number of observation posts needed to ascertain the exact number of stars in space?

b) Solve the same problem on a Flatland, the planar Universe.

6.(7-9). Consider n identical cars on a circular highway. The total quantity of fuel in all these cars is enough for one of them to cover the whole circle. Is it possible to find a car that can drive around the entire circle by borrowing fuel from other cars along the way for any arrangement of cars and distribution of fuel among them?

7.(8-10). Find non-negative integer solutions of the equation:

$$x^4 + 2x^3 + x^2 - 11x + 11 = y^2.$$

8.(8-10). On the numerical line, paint red all points that correspond to positive integers of the form $81x + 100y$, where x and y are positive integers; paint the remaining integers blue. Find a point on the line such that any two points symmetrical with respect to it are painted different colors.

9.(9-10). Integers $x_1, x_2, \dots, x_n; y_1, \dots, y_m$ satisfy the inequalities

$$1 < x_1 < x_2 < \dots < x_n < y_1 < \dots < y_m \quad \text{and} \quad x_1 + \dots + x_n > y_1 + \dots + y_m.$$

Prove that $x_1 x_2 \dots x_n > y_1 y_2 \dots y_m$.

10.(10). Prove that it is possible to cut any two polyhedrons of equal volume into several tetrahedrons of pair-wise equal volumes.

11.(8-9). Consider a square $ABCD$ and point O inside it. Prove that

$$135^\circ < \angle OAB + \angle OBC + \angle OCD + \angle ODA < 225^\circ.$$

12.(10). A) Given a finite set of n points not in the same straight line. For any two pairs of given points belonging to two different lines the intersection point of these lines also belongs to the set of given points. Prove that all points of the set but one lie on the same line.

B) Is it possible to draw n straight lines through point O in space so that *for any two of these lines there is a third straight line from the same set, which is perpendicular to the two lines* for (a) $n = 99$ or (b) $n = 100$?

(c) Point out all n for which there exists an arrangement of n lines satisfying the condition from heading B) and describe all possible arrangements of these lines in space.

13.(7-10). A pie is of the form of a square lamina. Two perpendicular straight lines cut the pie into four parts. Three of these parts weigh 200 g each. What is the weight of the pie?

14. (10). There are n point-size searchlights that illuminate angles (the vertex and the legs included) $\alpha_1, \alpha_2, \dots, \alpha_n$ on a plane. If these searchlights were placed at one point they would have illuminated the whole plane. Prove that for any n it is possible to permute the locations of searchlights (without rotating searchlights themselves) so that they would still illuminate the entire plane.

- 15.(8-9).** Consider convex quadrilateral $ABCD$ such that $AC = BD$, $\angle B = 2\angle C$, $\angle C + \angle D = 90^\circ$. Find angles $\angle B$ and $\angle D$ of the quadrilateral.
- 16.(9-10).** A) There are 9 points on the surface of a cube with edge 1. Prove that two of these 9 points are not farther than $\frac{\sqrt{3}}{2}$ from each other.
B) Can the surface of the cube with edge 1 have (a) 8 points and (b) 7 points so that the distance between any two of them is > 1 ?
- 17.(9-10).** a) The projections of a solid to two planes in space are circles. Prove that these circles are equal.
b) The projections of a convex n -gon to two non-parallel planes in space are regular n -gons. Prove that these projections are equal n -gons.
- 18.(8-9).** The sum of the digits in the decimal expression of 5^n is less than 10^{100} . Is the set of such positive integers n finite or infinite?
- 19.(7-9).** Prove that no digit is repeated 5 000 001 times in a row in the first 10 million digits of the decimal representation of $\sqrt{2}$ (the integer part included).
- 20.(7-9).** There are 1 000 airports in the land Shvambrania, and the distances between every two airports are distinct. Suppose an airplane departs from each airport and flies to the nearest airport. What greatest number of airplanes can land in an arbitrary airport if Shvambrania is a) a plane? b) a sphere?
- 21.(9-10).** Several spherical holes are made in a cheese cube. Prove that it is possible to cut the cheese into convex polyhedrons so that there is exactly one hole inside each of the polyhedrons.
- 22.(10).** Let $\sin \alpha = \frac{3}{5}$. Prove that $\sin 25\alpha = \frac{n}{5^{25}}$, where n is an integer not divisible by 5.
- 23.(7-10).** Three bulbs — one blue, one green, and one red — are somehow connected by wires to n switches. Each switch can be in one of three positions. For any position of all the switches exactly one bulb is turned on, but if all n switches are simultaneously flipped (each by its own of the 2 possible ways), another bulb is turned on. Prove that the color of the bulb which is turned on is determined by one fixed switch and does not depend in any way on the other switches.
- 24.(8-10).** A grasshopper hops on an infinite chessboard with squares of side 1 moving with each hop a distance of α to the right and β upwards. Prove that if numbers α , β and $\frac{\alpha}{\beta}$ are irrational, then the grasshopper will necessarily reach a black square.
- 25.(9-10).** Prove that $\tan \frac{3\pi}{11} + 4 \sin \frac{2\pi}{11} = \sqrt{11}$.
- 26.(8-10).** Solve in positive integers:
- $$520(xyzt + xy + xz + zt + 1) = 577(yzt + y + z).$$
- 27.(7-8).** Prove that if at all times at least one of ten uniformly functioning alarm clocks shows correct time, then at least one of them always shows correct time.
- 28.(9-10).** The space is divided into identical and identically *oriented* parallelepipeds.
a) Prove that for each parallelepiped at least 14 of the parallelepipeds have a common point with it.
b) What is the least number of parallelepipeds that have a common point with a given parallelepiped if the parallelepipeds are still identical but not equally oriented?
- 29.(8-10).** A triangular lamina of area 1 is cut into 4 parts (three triangles and 1 quadrilateral) by two straight cuts. Three parts have the same area. Find the area of every part.
- 30.(8-9).** Prove that if the arithmetic mean of the first $10^{10^{10}}$ digits in the decimal expression of $2 - \sqrt{2}$ is between $4\frac{1}{3}$ and $4\frac{2}{3}$, then the same is true for $\sqrt{2} - 1$.
- 31.(8-10).** Prove that at any given moment there is a point on the surface of the Sun (considered as a sphere) from which one can see not more than 3 planets (out of 9 known ones).
- 32.(7-9).** There are two containers: the first one has $1l$ of water in it, the second one is empty. We pour half of the water from the first container into the second one; then we pour one third of the water from the second container back into the first container; then we pour one fourth of the water from the first container into the second container, and so on. How much water is there in the first container after 12345 refills?
- 33.(9-10)*.** Prove that it is possible to arrange infinitely many squares with sides $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots, \frac{1}{n}, \dots$ inside a square with side $\frac{5}{6}$ so that they do not overlap, but it is impossible to do this in a square of a smaller side.
- 34.(8-10).** Given a wooden ball, a compass, and a piece of paper, draw on the paper a circle of radius equal to that of the ball. (It is allowed to draw circles on the ball.)
- 35.(9-10).** A square is divided in two ways into 100 parts of equal area. Prove that it is possible to select 100 points such that after each partition there is exactly one point in every part.

36.(8). Prove that the number

$$\frac{\left(2^4 + \frac{1}{4}\right) \left(4^4 + \frac{1}{4}\right) \left(6^4 + \frac{1}{4}\right) \left(8^4 + \frac{1}{4}\right) \left(10^4 + \frac{1}{4}\right) \left(12^4 + \frac{1}{4}\right)}{\left(1^4 + \frac{1}{4}\right) \left(3^4 + \frac{1}{4}\right) \left(5^4 + \frac{1}{4}\right) \left(7^4 + \frac{1}{4}\right) \left(9^4 + \frac{1}{4}\right) \left(11^4 + \frac{1}{4}\right)}$$

is an integer, and find the number by simplification without actual calculations.

37.(8-10). A table is entirely covered with 100 square tablecloths, A round hole burnt through the cover damaging each of the tablecloths. Each tablecloth would have covered the table but for the hole. Prove that some three of the tablecloths completely cover the table.

38.(8-10). Given 40 dice with the sum of the numbers on the opposite faces of each die equal to 7. Place the dice on one another to form a parallelepiped. Is it possible to rotate each die about its vertical axis so that the sums of the numbers on four lateral sides of the parallelepiped were equal?

39.(8-10). Consider two concentric circles, two parallel chords l and m tangent to the inner circle, point A on the outer circle between l and m ; tangents to the inner circle through A , and their intersection points C and D with the chords. Prove that the product $AC \cdot AD$ does not depend on the position of A .

40.(8-10). Prove that the difference between the numbers

$$1 - \frac{1}{2 + \frac{1}{3 + \frac{1}{4 + \dots \frac{1}{(n-1) + \frac{1}{n-1}}}}} \quad \text{and} \quad 1 - \frac{1}{2 + \frac{1}{3 + \frac{1}{4 + \dots \frac{1}{(n-1) + \frac{1}{n-1 + \frac{1}{n}}}}}}$$

is less than $\frac{1}{(n-1)!n!}$.

41.(8-9). Points A , B and C move uniformly along three circles in the same direction with the same angular velocity. How does the center of mass of triangle ABC move?

42.(8-9). Numbers 1, 2, 3, ..., 1974 are written on a blackboard. In one move we can erase any two numbers of the set and write in their place the absolute value of their difference. After 1973 moves one number is left. What number can it be?

43.(8-9). A and B are the tangent points of straight lines a and b and a circle. We selected point C on line a , and point D on line b . Segment AB meets segment CD at point M . Prove that $CA/CM = DB/DM$.

44.(8-10). Given $p+1$ distinct positive integers for a prime p , prove that among them there is a pair of numbers x and y such that the quotient after the division of the greater of these two numbers by $GCD(x, y)$ is not less than $p+1$.

45.(9-10). Prove that $\sum_{n=1}^k a_n \cos nx \geq -1$ for any x , then $a_1 + \dots + a_k \leq k$.

46.(9-10). Two people play a game in which one thinks a five-digit number consisting of 0's and 1's, and the other must guess it.

Guesser names a five-digit number consisting of 0's and 1's, and Thinker tells Guesser at how many places the digits of this number coincide with the corresponding digits of the one Thinker has in mind. Is it possible to guess the number in 3 guesses?

47.(9-10). A closed broken line is situated on the surface of a cube with edge 1. On each face of the cube there is at least one segment of the line. Prove that the length of the broken line is not less than $3\sqrt{2}$.

48.(9-10). Prove that for any integer $n \geq 2$ we have

$$n \left(\sqrt[n]{n+1} - 1 \right) \leq 1 + \frac{1}{2} + \dots + \frac{1}{n} \leq 1 + n \left(1 - \frac{1}{\sqrt[n]{n}} \right).$$

49.(7-10). On a white plane there sit a man and a black cat. The man is superstitious and does not want to cross the cat's path; the cat, full of spite, wants to move along a closed non-selfintersecting path so as to avoid the man and not give him a possibility to avoid the cat's path. Is it possible for the cat to circumvent the man within a finite length of time if its top speed is $\lambda > 1$ times that of the man? (The cat and the man may not be at the same point simultaneously.)

50.(8-10). There are 650 distinct points inside a disc of radius 16. Prove that there is an annulus with inner radius 2 and outer radius 3 on which lie at least 10 of the given points.

51.(8-10). Is there a positive integer n for which any rational number between 0 and 1 can be expressed in the form of the sum of n reciprocals of positive integers?

52.(8-10). A regular $2n$ -gon is inscribed in a regular $2k$ -gon, i.e., each vertex of the $2n$ -gon lies on the boundary of the $2k$ -gon. Prove that $2k$ is divisible by n .

53.(9-10). A cube contains a convex polyhedron M whose projection to each face of the cube covers the entire face. Prove that the volume of the polyhedron M is not less than one third of the volume of the cube.

54.(9-10). A city of the form of a square with side 10 km is divided into n^2 identical square blocks. The blocks are numbered from 1 to n^2 so that two blocks with consecutive numbers have a common side. Prove that a cyclist can find any block (s)he needs by riding not more than 100 km.

55.(8-10). In convex pentagon P_1 we drew all the diagonals. As a result P_1 split into 10 triangles and one pentagon, P_2 . Let S be the difference between the sum of the areas of the triangles adjacent to the sides of P_1 and the area of P_2 .

Let us perform the above operation (draw diagonals, etc.) with the inner pentagon P_2 ; let P_3 be its inner pentagon. Let s be the difference between the sum of the areas of the triangles adjacent to the sides of P_2 and the area of P_3 . Prove that $S > s$.

56.(8). Find the greatest number of vertices of a non-convex non-selfintersecting n -gon from which no inner diagonal can be drawn.

57.(8). At every integer point of the numerical line a positive integer is written. Between every two neighboring numbers we write their arithmetic mean and then erase the original numbers. We repeat this operation many times. It turns out that all numbers obtained after each step are positive integers. Is this sufficient to conclude that after some step all numbers will be equal?

58.(9-10). A $3 \times 3 \times 3$ cube is constructed of 27 cubic blocks with side 1. Each block is either white or black. Every hour a painter comes and white-washes all blocks with an even number of black neighbors, and paints black all the other blocks. Prove that eventually all blocks will be painted white.

59.(9-10). In space, there are n distinct points of equal mass. Consider sphere S_1 of radius 1 and with center at one of the given points. Let S_2 be the sphere of radius 1 (perhaps identical to S_1) with center at the center of mass of all the given points that are inside of S_1 . Let S_3 be the sphere of radius 1 (perhaps identical to S_2) with center at the center of mass of all the given points that are inside S_2 , etc. Prove that $S_k = S_{k+1} = \dots$ for some k .

60.(8). You are allowed to make two operations with ordered n -tuples of 0's and 1's: to change the first (left) digit and also to change the digit following the first (from the left) 1. Prove that such operations can turn any set into any other set.

61.(8). There are four equal circles O_1, O_2, O_3, O_4 inside a triangle such that circle O_1 is tangent to two sides of the triangle; circle O_2 is tangent to another pair of sides of the triangle; circle O_3 to the third pair of sides; and circle O_4 is tangent to the first three circles.

Prove that the center of O_4 lies on the same straight line as the centers of the circles inscribed in and circumscribed around the triangle.

62.(8-10)*. Two infinite (in both ways) non-selfintersecting broken lines are drawn on an infinite piece of graph paper. The segments of the broken lines are on the lines of the paper, and each broken line passes through all intersections of the grid of the paper. Must the broken lines have common segments?

63.(8-10). Denote the sum of the first n primes by S_n . Prove that there is a perfect square between S_n and S_{n+1} .

64.(8-10).** Consider square and an equilateral triangle are drawn on a plane. Prove that one of the distances between a vertex of the square and a vertex of the triangle is irrational.

65.(9-10).** A square town is divided into n^2 square blocks. The streets inside the town are two-way ones and the street skirting the town is a one-way one. A cyclist moves in the town in accordance with the following traffic rules: (s)he moves only along the right side of any street and does not turn left at intersections; on the one-way street that surrounds the town, (s)he moves so that all houses are on his/her right.

For what n can the cyclist ride through the whole town passing once each side of each street (and once the only side of the one-way street around the town)? Try to find the greatest possible set of such values of n .

66.(8). For an inscribed octagon $A_1A_2A_3A_4A_5A_6A_7A_8$ we have $A_1A_2 \parallel A_5A_6$, $A_2A_3 \parallel A_6A_7$, and $A_3A_4 \parallel A_7A_8$. Prove that $A_4A_5 = A_1A_8$.

67.(8-9). A square with side n is divided into n^2 square cells with sides 1. Can n^2 different numbers be written in the cells so that in any square whose sides coincide with the sides of the given $n \times n$ square or with the lines that divide it, the product of the numbers along one longest diagonal is equal to the product of the numbers along the other longest diagonal?

68.(8-10). A road is straight but not flat. Is it possible for three people to walk from points whose distances from the beginning of the road are 0, 1 and 2, respectively, to points whose distances from the

beginning of the road are 1000, 1001, and 1002, respectively, without passing one another and so that the last person sees the first one all the time and does not see the second person for a single moment? The heights of the people do not matter, i.e., a short one can see through a tall one.

69.(8). Denote the sum of the digits of a number N by $S(N)$. Prove that there is an infinite number of N 's that have no zeroes in their decimal expression and such that a) N is divisible by $S(N)$ or b) N is divisible by $S(N) + 1$.

70.(8-10). Prove that it is possible to construct a convex equiangular 1980-gon from segments of lengths 1, 2, 3, ..., 1980. Is the same true for a 1981-gon?

71.(9-10). Functions f and g defined on the real line are such that the equality

$$f(x - y) + f(x + y) = 2f(x)g(y)$$

holds for all x and y . Prove that if f is not identically equal to zero, then the values of g are not less than -1 for all y .

72.(9-10).¹ We have a *complete* graph: n points, every pair of which are connected by a segment. Each segment is painted either red or blue, and from any point one can get to any other point both along blue lines only and along red lines only.

Prove that there are four points among these n points such that the complete graph of these four points and the segments connecting them has the same property: one can get from any point to any other one both along blue lines only and along red lines only.

73.(9-10). Polynomial $P(x)$ is non-negative for all real x . Are there two polynomials $Q(x)$ and $R(x)$ such that $P(x) = Q(x)^2 + R(x)^2$?

74.(7-10). Find all integer solutions:

$$x^2 - 2y^2 = 6t^2 - 3z^2.$$

75.(10). A plane flew from town Γ_1 to town Γ_2 . During its entire flight it was seen from observation posts A and B hidden somewhere on segment $\Gamma_1\Gamma_2$. Prove that

a) there was one second during which the plane moved from some point X to a point Y of its trajectory in such a way that $\angle XAY = \angle XBY$;

b) statement a) is false if at least one of the observation posts is out of segment $\Gamma_1\Gamma_2$, no matter how close points A and B are from points Γ_1 and Γ_2 , respectively.

CONVENTION. We assume that the plane moves in space at a variable speed; the time of its flight is more than 1 sec; towns T_1 and T_2 , observation posts A and B , and the airplane are all points.

76.(9-10). A swimming pool has the form of a convex quadrilateral with trees growing in its vertices. Each tree casts a circular shadow with its center in respective vertex. It is known that the swimming pool is entirely in the shade. Prove that the shadow of some 3 trees entirely covers the triangle in whose vertices these trees grow.

77.(8-10). A colony of a finite number of bacteria lives on a straight line. Some bacteria may die at moments 1, 2, 3, ...; no new bacteria are ever born. Those and only those bacteria die for which there are no bacteria at a distance of 1 on their left and $\sqrt{2}$ on their right.

Can such a colony of bacteria live forever?

78.(9-10). Is there a finite set of points on a plane such that for each of the points there are at least 1000 other points of this set the distance between which and this point is exactly equal to 1?

79.(10). A dodecahedron with its vertices painted red is rolled over its edges on a plane, its vertices leaving red point marks. Prove that for any disc of any radius on the plane, it is possible to roll the dodecahedron so that some vertex leaves a red mark inside the disc.

80.(7-8). There are n boxes, some of which have n boxes inside them, some of which again have n boxes inside them, etc. There are altogether k boxes with other boxes inside them. What is the total number of boxes?

81.(8-10). Prove that if $x > 1$, $y > 1$, and $x^y + y^x = x^x + y^y$, then $x = y$.

82.(9-10). Prove that if a, b, c are the lengths of the lateral edges of a triangular pyramid and α, β, γ are the angles between the edges, then the volume of the pyramid is equal to

$$V = \frac{abc}{3} \sqrt{\sin \frac{\alpha + \beta + \gamma}{2} \sin \frac{\alpha - \beta + \gamma}{2} \sin \frac{\alpha + \beta - \gamma}{2} \sin \frac{-\alpha + \beta + \gamma}{2}}.$$

83.(10). a) Any section of a solid by a plane is a disc. Prove that the solid is a ball.

b)*** Any section of a solid is a polygon. Prove that the solid is a polyhedron.

¹Suggested by V.Gurvich for the selection competition to ??? in 1971.

84.(7-10). An organization committee of a Math Olympiad consists of 11 members. The problems for the Olympiad are kept in a strongbox. How many locks must the strongbox have and how many keys should every member of the committee have so that any six members can open the strongbox whereas no fewer group can do it?

85.(7-9). The hands of a clock are fixed but the dial can rotate. Prove that it is possible to turn the dial so that the clock shows a correct time between 12:00 p.m. and 1:00 p.m.

86.(8-10). Prove that if $0 < \alpha_1 < \alpha_2 < \dots < \alpha_n < \frac{\pi}{2}$, then

$$\tan \alpha_1 < \frac{\sin \alpha_1 + \dots + \sin \alpha_n}{\cos \alpha_1 + \dots + \cos \alpha_n} < \tan \alpha_n.$$

87.(7-8). Prove that if

$$\frac{1}{x+y+z} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z},$$

then two of the numbers x, y, z are equal in absolute value but have opposite signs.

88.(7-8). For which $n \in \mathbb{N}$ do there exist positive integers $k_1 < k_2 < \dots < k_n$ such that

$$\frac{1}{k_1} + \frac{1}{k_2} + \dots + \frac{1}{k_n} = 1?$$

89.(7-10). Prove that if the tips of the legs of a table are vertices of a square, then it is possible to place the table on an uneven floor so that the table does not rock, i.e. so that all four tips of the legs touch the floor.

90.(8-10). For $a, b, c > 0$ solve the system for unknowns x, y, z :

$$\begin{cases} \frac{a}{x} - \frac{b}{y} = c - xy, \\ \frac{b}{z} - \frac{c}{x} = a - xz, \\ \frac{c}{y} - \frac{a}{z} = b - yz. \end{cases}$$

91.(10). A sphere with center O is inscribed in tetrahedron $ABCD$. Prove that if $\angle ODC = 90^\circ$, then planes AOD and BOD are perpendicular.

92.(8-10). We write parentheses in the expression $x_1 : x_2 : x_3 : \dots : x_n$ with distinct x_i 's to indicate the order in which the numbers should be divided. The result is written in the form of the following fraction:

$$\frac{x_{i_1} x_{i_2} \dots x_{i_k}}{x_{j_1} x_{j_2} \dots x_{j_{n-k}}}.$$

How many distinct fractions of this kind is it possible to derive from the given expression by different arrangements of parentheses therein?

93.(7-9). Three soccer teams played the same number of matches with one another. Is it possible that the winner won the least number of matches while the team that took the last place won a maximal number of games?

94.(7-10). Prove that from the edges of an arbitrary tetrahedron it is possible to construct two triangles so that each edge is a side of one of the triangles.

95.(9-10). Consider three straight lines in space, each two of them skew and not all parallel to a plane. How many straight lines can intersect all three given lines?

96.(8-9). Twelve laces are used to make a net in the form of a cube with side of 10 cm. Inside the net is a spherical balloon. It is inflated so that the net fits tight on its surface. Find the radius of the inflated balloon.

97.(7-10). An entire rectangular map of Moscow lies on top of another similar map of a larger scale (the sides of the maps are not necessarily parallel). Prove that it is possible to puncture both maps with a pin so that the point of the puncture denotes the same point of the city on both maps.

98.(8-10). Is $2222^{5555} + 5555^{2222}$ divisible by 7?

99.(9-10). Three rods of equal lengths are used to construct a rigid spatial structure in which the rods do not touch one another but are connected by non-elastic threads fastened to their ends.

a) What least number of threads is necessary for this?

b)* What ratios of rod lengths and thread lengths make such a construction possible?

Hints to selected problems of Moscow mathematical circles

2. It is easy to see that there are no 1- and 2-digit examples. There are no 3-digit examples (for any base, not only decimal) either: indeed, it is clear that in the sum $\overline{abc} + \overline{cba}$ neither the first nor the last figure can be equal to either a or c .

A case-by-case checking shows that the least answer contains 5 digits.

REMARK. If the base were not 10, but, say, 9, there would have been 4-digit examples, say, $2563_9 + 3652_9 = 6325_9$. Similar examples exist for any base divisible by 3 starting with 6.

4. Prove that the sum of these 14 numbers either is equal to 14 (if all vertices are labeled by 1's) or differs from 14 by a multiple of 4, i.e. can be $10, 6, 2, -2, \dots$. It suffices to prove that by changing the sign of one of the units at a vertex we alter the sum by a multiple of 4.

9. Make use of the fact that if $1 < z < t$ then $z + t < zt$.

10. First, divide both polyhedrons into arbitrary tetrahedrons (e.g., one can first divide a polyhedron into pyramids by connecting its inner point with the vertices). Then, selecting the smallest of the tetrahedrons obtained, one should cut a tetrahedron of the same size from one of the remaining tetrahedrons. Further, apply the induction on the number of tetrahedrons.

11. Let M be the center of square $ABCD$ and let O lie, for example, in the triangle ABM . Now, prove that $\angle OAB + \angle OCD \leq 90^\circ$. The inequality $\angle ODA < 45^\circ$ being taken into account, it suffices to prove the estimate from above.

The estimate from below is derived in the same way from $\angle OBC + \angle ODA \geq 90^\circ$.

18. If this set is infinite, the decimal representation of 5^n would contain too long sequences of zeros for a large n .

21. Define the distance from a point outside a sphere to the sphere as the length of the tangent from the point to this sphere. For each spherical hole, consider the set of all points "distanced" from the sphere of the hole not farther than from any other hole. It is easy to prove that for two spherical holes we thus get two half-spaces. In the general case, each of the sets obtained is a convex polyhedron containing a sphere. All polyhedrons adjoining each other entirely fill up the cube.

24. First, find an integer m such that the numbers¹ $\{m\alpha\}$ and $\{m\beta\}$ are both small (say, smaller than 0.01) but their ratio is not less than 2.

27. If the alarm-clock ticks uniformly but generally shows wrong time, then during, say, an hour there is only a finite number of moments when it indicates a right time.

28. a) It is easy to demonstrate that the parallelepipedal lattice can be replaced by that of cubes and the solution of the problem will not change.

Circumscribe the ball round each of 8 vertices of the given cube (from cubic lattice) centered at each vertex; let the volume of each ball be equal to 1. The sum of the balls' volumes is equal to 8 and the volume of the balls inside the cube is equal to $8 \times \frac{1}{8} = 1$. So the remaining $8 - 1 = 7$ volume units fall on the neighboring cubes. Let us prove that each neighboring cube has not more than $\frac{1}{2}$ of the volume unit, and, therefore, there number is not less than 14.

Indeed, if the vertex of the given cube lies inside the face of the neighboring cube, this volume is $\frac{1}{2}$ of the volume unit (the neighboring cube does not touch other balls).

If the vertex of the cube lies on an edge of the neighboring cube, one more vertex can lie on its edges and the total volume does not exceed $2 \times \frac{1}{4} = \frac{1}{2}$ of the volume unit.

Finally, if the vertex of our cube is also the vertex of the neighboring cube, this cube may touch maximum 4 more vertices of our cube and this gives a volume of $4 \times \frac{1}{8} = \frac{1}{2}$ of the volume unit.

b)² See Fig. 99.

32. After any odd number of refills both containers have the same amount of water.

36. Make use of the identity $\left(n^4 + \frac{1}{4}\right) = \left(n^2 + n + \frac{1}{2}\right) \left(n^2 - n + \frac{1}{2}\right)$.

37. Apply Helly's theorem: *If any 3 of n given convex figures (e.g., discs) on the plane have a common point, then this point belongs to all of them.*

39. Prove that $AC \cdot AD = R^2$, where R is the radius of the exterior circle. This follows from the fact that $\triangle OAC \sim \triangle OAD$.

¹Here $\{x\}$ is the fractional part of x .

²We thank M. Urakov who suggested this extension.

FIGURE 99. (Hint A28)

42. Apply induction with the following hypothesis: the procedure described in the problem allows one to get from $1, 2, \dots, 2k$ any number 0 to $2k$ whose parity coincides with that of k .

47. The length of the projection of the given broken line to any edge is > 2 . Therefore, the sum of these projections is ≥ 6 . Now, prove that every chain of the broken line is not more than $\sqrt{2}$ times shorter than the sum of its projections. Make use of the fact that one of the projections of any chain is always equal to 0 .

48. Make use of the Cauchy inequality for the arithmetic mean and the geometric mean. For the left inequality make use of the fact that

$$\sqrt[n]{n+1} = \sqrt[n]{\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \dots \cdot \frac{n+1}{n}} < \frac{1}{n} \left(\frac{2}{1} + \frac{3}{2} + \dots + \frac{n+1}{n} \right).$$

Find on your own a similar inequality for the right side of the problem.

52. Prove first that the centers of both polygons coincide.

54. The cyclist should act as follows: First, (s)he should divide the square 10×10 into 4 squares with the midlines and find out which of the squares thus obtained contains the city block (s)he is looking for (assuming that the numbers of adjacent blocks differ by 1); then divide this quarter of the square in a similar way and select the 16-th part (*2-nd order square*) which contains the block (s)he wants, etc., see Fig. 100.

To evaluate in each particular case the longest path the cyclist takes requires to sum the number of stages to the appropriate square of the highest order.

The first stage takes 25 km (or a bit more if n is odd); each next stage takes one half of the preceding one.

56. Prove that of two neighboring vertices, only one can satisfy the condition of the problem.

57. First, prove that if not all numbers are equal and m is the smallest of them (it may occur not once but many, even an infinite, number of times) then eventually all numbers on any segment of finite length become greater than m .

58. Let us write 1 on each white cube and -1 on each black cube. The painter's performance is an operation that replaces every number by the product of its neighboring ones. It is easy to deduce that it suffices to consider the case of one black and 26 white cubes; the general case is obtained from this one by the above "multiplication". It remains to consider 4 variants (deal with them on your own): when the black cube is in the corner; on the edge; in the center on the face; in the center of the cube.

59. Consider the function $f(X) = \sum (1 - |A_i X|^2)$, where the sum runs over the terms with A_i contained in the ball of radius 1 centered at X . Prove that the value of f increases when we replace X with the center of mass of A_i 's.

60. Prove by induction on n that any set can be reduced by the described operations to any of the following two forms: 000...00 or 000...01.

61. First, notice that the center O_4 of the circle tangent to the given circles with centers at O_1, O_2, O_3 is the same as that of the circle circumscribed about triangle $O_1O_2O_3$ and their radii differ by the radius of one of the given circles, O_1, O_2 , or O_3 . The intersection point of the bisectors of $\triangle ABC$ is the same as that of $\triangle O_1O_2O_3$ (the latter are continuations of the former), i.e., point O is the center of the circles inscribed into $\triangle ABC$ and $\triangle O_1O_2O_3$. At the same time, O is the center of a homothety for the homothetic triangles $\triangle ABC$ and $\triangle O_1O_2O_3$ (since $AB \parallel O_1O_2, BC \parallel O_2O_3, AC \parallel O_1O_3$).

FIGURE 100. (Hint A54)

FIGURE 101. (Hint A56)

Let point O' be the center of the circle circumscribed about $\triangle ABC$. It is easy to see that the homothety with the center at O which sends $\triangle O_1O_2O_3$ into $\triangle ABC$ sends O_4 to O' . Hence, O_4 belongs to $O'O$, Q.E.D.

63. Let p_n be the n -th prime. Make use of the two facts: that S_n is (for $n > 4$) less than the sum of the first n odd numbers (which is equal to n^2) and $p_{n+1} > p_n + 2$.

67. First, prove that the condition of the problem will be satisfied if and only if the product of the numbers in two opposite corners of any square is equal to the product of the numbers in the other two corners. The simplest way to proceed now is to place the number p_iq_j in the square at the intersection point of the i -th row and the j -th column, where p_1 and $p_2, \dots, p_n, q_1, \dots, q_n$ are numbers arbitrary except that all products p_iq_j are distinct, e.g. take $2n$ distinct primes.

70. Make sure first that an equiangular hexagon can be composed of segments of length $k+1, k+2, \dots, k+6$ (for any k), then the 1980-gon is composed of several hexagons. (Fig. 102 shows how it can be done for a 12-gon.)

This construction is possible due to the fact that $\cos 60^\circ = \frac{1}{2}$ is a rational number. Since $\cos \frac{360^\circ}{n}$ is irrational for $n = 1981$ and for the nontrivial divisors of 1981, the problem cannot be solved for the 1981-gon.

72. Use induction on n .

FIGURE 102. (Hint A70)

91. Draw the plane π perpendicular to OD through the center O of the sphere. Then make use of the facts that DC is parallel to π and π intersects lines parallel to DC on the planes of the faces ADC and BDC .

92. Prove by induction that the result may be any fraction with x_1 in the numerator and x_2 in the denominator.

The induction should start with $n = 3$. It should be born in mind that if $(x_n : x_{n+1})$ is substituted for x_n in the expression $x_1 : x_2 : \dots : x_n$, the final result will be that x_{n+1} is in the numerator if x_n was in the denominator and vice versa. But if $(P : x_n)$ is replaced by $((P : x_n) : x_{n+1})$ in the expression $(x_1 : \dots : (P : x_n))$, where P is a bracket or just a letter x_{n-1} , then x_{n+1} takes the same place as x_n .

95. We draw all kinds of planes through skew line No. 2, each of which intersects skew lines No. 1 and No. 3 at two points. By connecting the latter we get a straight line l that intersects all given lines Nos. 1, 2 and 3. (It is not difficult to show that only one of such lines can be parallel to line No. 2.)

Answers to selected problems of Moscow mathematical circles

1. a) $(1 + 2 + \dots + 9) \cdot (6 \cdot 10^5) = 27 \cdot 10^6$; b) $6 \cdot 10^5$.
2. $12897 + 79821 = 92718$.
3. There are.
4. No. It can not.
6. Yes.
7. $(x, y) = (1, 2)$ or $(2, 5)$.
8. $\frac{1}{2}(8100 + 181)$.
13. 800 g.
15. $\angle B = 60^\circ$, $\angle D = 150^\circ$.
16. A) ??
B) 7 points can be arranged in the way required but 8 points can not.
18. This set is finite.
29. There are two cases depicted on Fig. 103: in case b) $S_1 = S_2 = S_3 = \frac{1}{6}$, $S_4 = \frac{1}{2}$; in case c)
 $S_1 = S_2 = S_3 = \frac{\sqrt{5}-1}{4}$, $S_4 = \frac{7-3\sqrt{5}}{4}$.

FIGURE 103. (Answ. A29)

36. 313.
38. No, this is not always possible.
42. Any odd number between 1 and 1973.
46. No.
49. Yes.
51. No, such n does not exist.
56. $\left\lfloor \frac{n}{2} \right\rfloor$; for an example see Fig. 104.
57. No.
62. No, see Fig. 105.
65. For any n .
67. Yes.

FIGURE 104. (Answ. A56)

FIGURE 105. (Answ. A62)

68. Yes, it is possible, see Fig. 106. The three persons should change their respective coordinates in the following manner:

- a) $2 \rightarrow 1002$ (visible from 0);
- b) $0 \rightarrow 0.5$;
- c) $1 \rightarrow 1001$ (not visible from 0.5);
- d) $0.5 \rightarrow 1000$ (visible from 1002, not visible from 1001).

FIGURE 106. (Answ. A68)

73. Certainly.

77. No.

78. Yes.

80. $(k+1)n$ boxes.

84. The number of locks is $\binom{11}{5} = 462$ and the number of keys for each member of the organizing committee is $\binom{10}{5} = 252$.

88. Any $n \geq 3$.

92. 2^{n-2} .

93. Yes; an example is shown in Table:

	I	II	III	Total	Score
I	XXXXX	+1 =6 -0	+2 =3 -2	+3 =9 -2	15
II	+0 =6 -1	XXXXX	+4 =0 -3	+4 =6 -4	14
III	+2 =3 -2	+3 =0 -4	XXXXX	+5 =3 -6	13

95. Infinitely many.

98. Yes.

Solutions to selected problems of Moscow mathematical circles

1. The given number is “built” of numbers 1 to 999999. The sum of digits, as well as the number of digits 7, does not vary if we insert several zeroes, say, as follows : 000000 000001 000002 ... 999999. Now, it is clear that we have written $6 \cdot 10^6$ digits which constitute all possible combinations of 6 digits. Therefore, all digits are encountered the same number of times, namely $(6 \cdot 10^6)710$. this immediately implies the answer.

3. If $(\sqrt{2})^{\sqrt{2}}$ is rational, take $y = x = \sqrt{2}$. Otherwise take $x = (\sqrt{2})^{\sqrt{2}}$, $y = \sqrt{2}$, then $x^y = \left((\sqrt{2})^{\sqrt{2}} \right)^{\sqrt{2}} = (\sqrt{2})^2 = 2$.

4. Let all vertices be labelled by 1; then the sum equals 14. By changing the number at a vertex we change three more numbers (on three adjacent faces), so the sum diminishes by 8 and becomes 6.

Now, let us change the numbers at one more vertex. Again, this induces a simultaneous change of 3 more numbers and one easily sees that the sum differs by $\pm 8, \pm 4$ or 0. Namely, if four 1's turn into -1 's, then the sum diminishes by 8; if three 1's and one -1 change, the sum diminishes by 4, and so on.

Therefore, the sum may differ from 14 by a multiple of -4 , i.e., it can be 10, 6, 2, -2 , ... So the sum can not be equal to 0 or 7.

5. In this problem it is tacitly assumed that a star and an observation post are just points.

Observe that the answer for the 3-dimensional space differs from that for the 2-dimensional space (plane): *two* observation points are enough for the former and *three* for the latter.

a) Since one observation point is obviously insufficient, choose point A at random and consider all rays, starting from A , on which lie all stars visible from A . (“Consider all ... ” means “plot the rays and indicate their position in space relative to a preselected system of coordinates”).

Now, consider all possible planes π_1, π_2, \dots , drawn through each pair of the rays. Let B be an arbitrary point that does not belong to any of the planes π_1, π_2, \dots . Let us prove that B is the required observation post since ALL stars in the sky can be seen from there, i.e., the stars visible from B do not hide behind each other.

Indeed, all stars lie on straight lines that connect pairs of stars. Thus, all stars are in planes π_1, π_2, \dots . Since point B does not belong to any of these planes by construction, B does not belong to any of the lines connecting the stars. But the stars can hide behind each other only from an observer located on these lines. Thus, two posts suffice.

b) The reasoning in a) does not apply since there is no point B outside the plane. Two observation posts are not enough because for any chosen observation post B stars may happen to be on a ray coming from point B and crossing the rays coming from the first point, A , to the stars; their intersection points may happen to be stars.

So, select at random two points A and B from which not all stars may be visible. But all stars are sure to lie at some of the points where rays connecting A with stars intersect the rays connecting B with stars.

It remains to draw all possible lines through all pairs of the intersection points of the two bunches of rays with vertices at A and B , respectively, and to select a point, C , not belonging to either of these lines or any of the rays from the bunches starting from A or B . As is clear from this construction, the point C is the desired one.

6. Proof: by induction. If there is only one car, there is nothing to prove.

Suppose the statement is already proved for $n - 1$ cars and consider n cars. Clearly, at least one of the cars (call it A) has enough gas to drive to the next car, B . Remove B from the road and add its petrol to A . Now there are $n - 1$ cars on the road with the same quantity of gas and by the inductive hypothesis there exists a car C which can run the whole length of the road. Notice that the same car C can run the whole length of the road also in the initial situation when car B is present on the road.

7. For $x = 3$ there are no solutions, since the left hand side is equal to 122 which is not a perfect square.

Let us show that for $x > 4$ there are no solutions either. Observe that to find the minimal positive value of $z = R^2 - t^2$ for a fixed R and variable integer t , one has to look at the graph of this function to deduce that the minimum is attained at $t = R - 1$.

Let us now rewrite the equation in the form

$$(x^2 + x)^2 - y^2 = 11(x - 1). \quad (*)$$

By the observation above, the left hand side of (*) takes the least positive values for a fixed x if $y = (x^2 + x) - 1$. But then it is equal to $2x^2 + 2x - 1$ which is greater than $11(x - 1)$ for $x > 4$ as is easy to verify by setting

$t = x - 1$. Therefore, if the left hand side of (*) is positive, then it is greater than the right hand side. Therefore, the solutions indicated above are the only ones.

8. First, let us prove that all numbers greater than 8100 are painted red. Indeed, let $8100 < A = 81k + 100l$. By Euclid's algorithm, any integer can be represented in this form with integer, though not necessarily positive, k and l . At least one of the numbers k and l , say, k , is positive. From all representations of A in the above form select the one for which k is the least positive. Then $k < 100$ (otherwise there would have existed a representation $(k - 100, l + 81)$), therefore, $81k < 8100$. But then $100l > 0$ and $l > 0$, as was required.

As is not difficult to figure out, the number 8100 is painted blue; hence, this is the right-most of the blue points. On the other hand, it is clear that the left-most of the red points is 181. Therefore, it is clear that the point whose existence is claimed in the problem should equal to $\frac{1}{2}(8100 + 181)$ and numbers A and B are symmetric with respect to it if $A + B = 8281$.

It remains to prove that of two such numbers one is necessarily red and another one blue. Let us write again $A = 81k + 100l$ and $B = 81m + 100n$. Assume that k and m are the least positive numbers for which such a representation exists, i.e., $0 < k, m < 100$. Then

$$8281 = 81(k + m) + 100(l + n)$$

and $0 < k + m \leq 200$.

But it is not difficult to verify that there exists only two representations of 8281 in the form $81x + 100y$ so that $0 < x < 200$. These representations are: $x = 1, y = 82$ and $x = 101, y = 1$.

In the first case $k + m = x = 1$ which contradicts the condition $k, m > 0$. Thus, the second case holds. But then $l + n = 1$ and therefore, one of these numbers is positive and the other one is not. This directly implies that one of the numbers A, B is red. We leave it to the reader to establish that the other number is blue.

12. A) Draw all possible straight lines through all pairs of points in the set. Denote straight line l containing points A and B by AB . We can now demonstrate that at least three points of the set lie on at least one of the drawn lines.

Indeed, if A, B, C , and D are four points of the set which do not coincide, and M — the intersection point of AB and CD — also belongs to the set, then A, B, C are either on the same straight line and then M coincides with C (see Fig. 107 a)) or three points A, B, M and, respectively, C, D, M already lie on straight lines AB and CD ; see Fig. 107 b).

So let us choose a line l containing at least three points of the set and prove that all points of the set but one lie on that line. Assume the contrary: let A and B lie outside l . Let us prove then that the set contains an infinite number of points: contradiction.

To this end denote the intersection point of l with AB by C_1 . By the hypothesis it belongs to the set. (In what follows we will remember that all intersection points of the lines under consideration belong to the set.) Line l was said to have also points C_2 and C_3 of the set, see Fig. 107 c). Denote the intersection point of AC_3 with BC_2 by X_1 (AC_3 and BC_2 are not parallel), X_1 not lying in l . Lines AC_2 and C_1X_1 meet at point X_2 , also outside l . Then BX_2 intersects l at point C_4 which does not coincide with either C_1, C_2 , or C_3 (since BC_4 does not coincide with either BC_1 or BC_3).

Further on, AC_4 meets C_1X_1 at X_3 and then BX_3 meets l at point C_5 distinct from C_1, \dots, C_4 .

Let us continue the same operation: if $C_k \in l$ is already plotted, lines AC_k and C_1X_1 meet at a point X_{k-1} distinct from X_1, \dots, X_{k-2} and then BX_{k-1} intersects l at point C_{k+1} distinct from C_1, \dots, C_k . The process may go on *ad infinitum* and an infinite number of points from the set will appear on l (and we did not yet consider other intersection points of the lines!). Since the given set is a finite one, the contradiction proves that our assumption was wrong.

FIGURE 107. (Sol. A12)

B) Let us start solution with heading (c). We will give two solutions related to different branches of geometry (see Remark below).

FIRST SOLUTION. Draw plane π which does not pass through point O and is not parallel to either of the straight lines l_1, \dots, l_n of the set (a “generic” plane). Denote n intersection points of these lines with π by A_1, \dots, A_n , respectively. Hereafter we will denote the plane containing the straight lines l_i and l_j by $l_i l_j$ and denote the line through points A_i and A_j by $A_i A_j$.

Now, we can prove that the set of points $A_1, \dots, A_n \in \pi$ satisfies the condition of Part A) of the solution. Indeed, let $A_i A_j$ and $A_k A_p$ be two distinct straight lines. We can prove that they meet at one of the points A_s of the set (generally, A_s may coincide with either of the points A_i, A_j, A_k , and A_p).

To this end consider planes $l_i l_j$ and $l_k l_p$. These planes intersect because they contain point O . Denote by l the straight line of their intersection. Let us prove that l is a straight line from our set. Indeed, denote a line from the set perpendicular to l_i and l_j (there is such a line by hypothesis) by L_1 and a line from the set perpendicular to l_k and l_p by L_2 . Then l is perpendicular to plane $L_1 L_2$ because both L_1 and L_2 are perpendicular to l .

On the other hand, by the hypothesis a straight line perpendicular to plane $L_1 L_2$ must belong to the set l_1, \dots, l_n . Thus, l is a straight line from the set, Q.E.D.

If $l = l_s$, then point A_s which belongs to l_s must also belong to line $A_i A_j$ (since l belongs to $l_i l_j$) and to line $A_k A_p$ (because l belongs to plane $l_k l_p$). Thereby we have proved that lines $A_i A_j$ and $A_k A_p$ meet at point A_s from the set.

But then it follows from heading A) that $n - 1$ points of the set lie on the same line (let them be A_1, \dots, A_{n-1}) and the remaining point A_n lies outside this line.

Consequently, the lines l_1, \dots, l_{n-1} lie on the same plane P while l_n is outside it. How then is l_n arranged relative to plane P ?

We see that $l_n \perp P$, since there is a line from the set perpendicular to the lines l_1 and l_2 , and l_n is the only line from the set perpendicular to the plane $l_1 l_2$.

Finally, taking l_n and an arbitrary line l_i , $l_i \subset P$, we deduce from the hypothesis that among the remaining lines of the set lying in plane P there exists a line l_j perpendicular to both l_i and l_n . Hence, for each line L from the set, $L \subset P$, there exists a line M from the set such that $M \subset P$ and $M \perp L$. Thus, all lines $l_1, \dots, l_n \subset P$ can be divided into pairs of pair-wise perpendicular lines. Hence, $n - 1$ is even; thus, n is odd ($n \geq 3$).

But for any odd n the desired arrangement of the lines exists and can be uniquely described as the union of a line, l , with $n - 1$ lines lying in the plane perpendicular to l ; moreover, the $n - 1$ lines consist of $\frac{n-1}{2}$ pairs of mutually perpendicular lines.

Therefore, the answers to (a) and (b) are: 99 straight lines can be drawn in the way required and 100 lines can not.

SECOND SOLUTION. Let us prove that all straight lines save one lie on one plane. (The remaining part of this solution does not differ from the first solution). For $n = 3$ the statement is clear. For $n > 3$ it is impossible that all lines are pair-wise perpendicular. Take two non-perpendicular lines, x and y . Let their common perpendicular be the z -axis. Two more lines should lie in xy -plane: the common perpendiculars to the z -axis and the original lines.

Let us prove that all other lines lie in xy -plane. Assume the contrary and among all lines select the one, l , forming the least angle with the z -axis. Of 4 lines lying on xy -plane, select one, m , not in plane zl and not perpendicular to zl -plane. Since lines m and l are not perpendicular, a line p lying on ml -plane and perpendicular to m also belongs to our set.

It remains to observe that line p is the image of the orthogonal projection of the z -axis to ml -plane; hence, the angle between z and p is smaller than the angle between z and any other line lying in ml -plane. In particular, it is smaller than the angle between z and l . Contradiction. Q.E.D.

REMARK. Problems A) and B) are statements of the so-called “geometry of order” or “geometry of position” (another name is “descriptive geometry”) where the main idea is that of a “position between” (e.g. a segment is a set of points lying between two given points, etc.) and the idea of an “arrangement in a certain order”. The solution of A) and the first solution of B) are in the spirit of this geometry. The second solution uses the notion of distance (angle) and is related to the metric geometry.

13. To demonstrate that the total weight is equal to 800 g, let us prove that both straight lines pass through the center of the square. If the lines do not meet at the center, let us translate them parallelly so that they would. After this translation one of the pieces increases in weight and the opposite piece decreases and all four pieces become equal figures. Contradiction.

14. This problem generalizes Problem 30.2.7.5. We will describe a generalization of the three-dimensional analogue of this problem (cf. Problem 30.2.10.4) at the end of the solution below.

First of all, we outline the idea of the solution. Let us randomly distribute the searchlights over given points M_1, M_2, \dots, M_n . If not the whole plane is illuminated we will construct another, “improved”, arrangement of the searchlights, the “quality” of the arrangement being evaluated with a numerical function f .

Since the number of the arrangements of searchlights over the points M_1, M_2, \dots, M_n is finite, we will automatically light up the entire plane by taking the arrangement for which the value of f is *maximal*. Indeed, if not the whole plane were illuminated in this case we could still improve the arrangement: contradiction.

FIGURE 108. (Sol. A14)

Let us start carrying out this plan. Let us transport all searchlights to one point O and draw a convex n -gon $A_1A_2 \dots A_n$ whose sides are lighted up by the searchlights¹, see Fig. 108; let $\angle A_iOA_{i+1} = \alpha_i$, $i = 1, \dots, n$. Drop the perpendiculars OH_i from O to the sides of the polygon or to their extensions. Let $|OH_i| = h_i$. Consider the vectors

$$\vec{e}_i = \frac{1}{h_i^2} \cdot \overrightarrow{OH_i} \quad \text{for } i = 1, \dots, n.$$

Clearly, $|\vec{e}_i| = \frac{1}{h_i}$, see Fig. 108 b).

Let us attach each vector \vec{e}_i to its respective searchlight α_i . Then the arrangement of the searchlights over the points M_1, \dots, M_n corresponds to the distribution of the vectors $\vec{e}_1, \dots, \vec{e}_n$ over these points (we arrange the searchlights together with the vectors as solid bodies).

To justify the appearance of the strange vectors \vec{e}_i we need the following geometric fact:

Lemma 1. *Suppose the searchlight α_p placed at point M illuminates a point N while the searchlight α_q placed at M does not illuminate N . Then $\overrightarrow{MN} \cdot \vec{e}_p > \overrightarrow{MN} \cdot \vec{e}_q$, see Fig. 108 c).*

(Hereafter a “ \cdot ” means the inner product of vectors; the i -th searchlight is denoted by α_i — the angle it illuminates indexed by its number).

PROOF. Let $\overrightarrow{MN} = \vec{v}$. Draw the vector $\overrightarrow{OP} = \vec{v}$ with O as its initial point, see Fig. 108 d).

The ray $[OP)$ intersects the p -th side of the polygon $A_1 \dots A_n$ at a point K and does not intersect the q -th side because the searchlight α_p lights up N while α_q does not.

There are two possibilities:

- 1) $[OP)$ does not intersect the straight line on which the q -th side lies;
- 2) $[OP)$ intersects this line at a point R .

In case 1), Lemma 1 is obvious since $\vec{v} \cdot \vec{e}_p > 0$ and $\vec{v} \cdot \vec{e}_q \leq 0$.

In case 2), the inequality $|OK| < |OR|$ is satisfied and, therefore,

$$\begin{aligned} \vec{v} \cdot \vec{e}_p &= |\overrightarrow{OP}| |\vec{e}_p| \cos(\vec{v}, \vec{e}_p) = |\overrightarrow{OP}| \frac{1}{|\overrightarrow{OH_p}|} \cos(\overrightarrow{OH_p}, \overrightarrow{OK}) \\ &= \frac{|\overrightarrow{OP}|}{|\overrightarrow{OK}|} > \frac{|\overrightarrow{OP}|}{|\overrightarrow{OR}|} = |\overrightarrow{OP}| \frac{1}{|\overrightarrow{OH_q}|} \cos(\overrightarrow{OH_q}, \overrightarrow{OK}) = |\overrightarrow{OP}| |\vec{e}_p| \cos(\vec{v}, \vec{e}_q) = \cos(\vec{v}, \vec{e}_q). \quad \square \end{aligned}$$

Now, let us introduce a function $f_\sigma(N)$ that depends on the point N of the plane and on the distribution σ of searchlights over the points M_1, \dots, M_n :

$$f_\sigma(N) = \overrightarrow{M_1N} \cdot \vec{e}_{\sigma(1)} + \overrightarrow{M_2N} \cdot \vec{e}_{\sigma(2)} + \dots + \overrightarrow{M_nN} \cdot \vec{e}_{\sigma(n)}, \quad (*)$$

¹If we leave out the 3-dimensional generalization for better days, the following solution can be considerably simplified by assuming that all OA_i are of unit length.

where $\sigma(i)$ is the number of the searchlight placed at point M_i .

Lemma 2. *If point N is not lighted up under the arrangement σ of the searchlights, there is an arrangement τ of searchlights such that $f_\tau(N) > f_\sigma(N)$.*

PROOF. First, we give an algorithm to improve the arrangement σ .

ALGORITHM. Let all searchlights α_i first stand at points M_i with the same index. Let α_1 not light up N . Remove it for a while and place at point M_1 the searchlight which does illuminate N . (Explain yourselves why such a searchlight will certainly be found.)

Let it be α_2 . Now, there is nothing at point M_2 , where α_2 used to be. Out of the remaining searchlights, place at M_2 the one which will light N . Let it be α_3 . Then we place the searchlight α_4 at M_3 to light N , the searchlight α_5 at M_4 , and so on, until we have a cycle, see Fig. 109 e).

FIGURE 109. (Sol. A14)

This means that the searchlight α_{k+1} is moved over to M_k , the searchlight α_{k+2} is moved to M_{k+1} , etc., and α_k to M_s , where $s > k$. Bring back all searchlights $\alpha_1, \dots, \alpha_{k-1}$ which did not get into the cycle to their initial points M_1, M_2, \dots, M_{k-1} .

The result of the application of the algorithm is that N is lighted up by *each* searchlight $\alpha_k, \alpha_{k+1}, \dots, \alpha_s$, where $s \geq k + 1$, because N was not lighted up before.

Let us see how $f(N)$ changed under the application of the algorithm. The summands with the index of M_i not equal to $k, k + 1, \dots, s$ in the sum (*) did not change, whereas, as follows from Lemma 1, the summands with the indices $k, k + 1, \dots, s$ increased. Q.E.D.

However, f_σ cannot be directly used yet to evaluate the “quality” of the arrangement σ since it depends on N . The situation is saved by the wonderful

Lemma 3. *For any two points N_1 and N_2 , the difference $f_\sigma(N_1) - f_\sigma(N_2)$ does not depend on the arrangement of searchlights.*

PROOF. Observe that the sum $\sum_{i=1}^n \overrightarrow{e_{\sigma(i)}}$ does not depend on the choice of an arrangement; therefore,

$$\begin{aligned} f_\sigma(N_1) - f_\sigma(N_2) &= \sum_{i=1}^n \overrightarrow{M_{\sigma(i)}N_1} \cdot \overrightarrow{e_{\sigma(i)}} - \sum_{i=1}^n \overrightarrow{M_{\sigma(i)}N_2} \cdot \overrightarrow{e_{\sigma(i)}} \\ &= \sum_{i=1}^n \left(\overrightarrow{M_{\sigma(i)}N_1} - \overrightarrow{M_{\sigma(i)}N_2} \right) \cdot \overrightarrow{e_{\sigma(i)}} = \sum_{i=1}^n \overrightarrow{N_2N_1} \cdot \overrightarrow{e_{\sigma(i)}} = \overrightarrow{N_2N_1} \cdot \sum_{i=1}^n \overrightarrow{e_{\sigma(i)}} = \text{const.} \quad \text{Q.E.D.} \end{aligned}$$

Let us prove that the function

$$f_\sigma = f_\sigma(N) + \overrightarrow{NN_0} \cdot \sum_{i=1}^n \overrightarrow{e_{\sigma(i)}} = \sum_{i=1}^n \overrightarrow{M_{\sigma(i)}N_0} \cdot \overrightarrow{e_{\sigma(i)}},$$

where N_0 is an arbitrary fixed point in the plane, is the desired one.

Indeed, the value of f_σ does not depend on N and increases under any rearrangement described in the algorithm.

Thus, we have found the “quality function”, f_σ , which only depends on the arrangement σ of the searchlights and demonstrated how to increase its value if a point in the plane is not lighted up. Since f_ω as the function of its index, the arrangement (for a fixed arrangement it is a constant as we just proved), attains its maximum for an arrangement ω , the entire plane is lighted up in this case.

EXTENSION. (THREE-DIMENSIONAL GENERALIZATION). Let there be given a convex polyhedron with faces $\Gamma_1, \dots, \Gamma_n$ and a point O inside, as well as n arbitrary points M_1, \dots, M_n in space. Each face Γ_i

defines a polyhedral angle α_i at which this face is seen from O , see Fig. 109 f). There are n searchlights at O , the i -th searchlight lighting up the polyhedral angle α_i ($1 < i < n$); thus the entire space is lighted up from O .

Prove that the searchlights can be moved over to points M_1, \dots, M_n so that the whole space will be still lighted up. The solution of the problem almost literally repeats that of the flat version.

15. Denote: $\angle ABD = \alpha$, then $\angle ACD = \frac{\alpha}{2}$, $\angle ADB = 90^\circ - \frac{\alpha}{2}$, see Fig. 110.

FIGURE 110. (Sol. A15)

From $\triangle ABD$ we have

$$\angle BAD = 180^\circ - (\alpha + 90^\circ - \frac{\alpha}{2}) = 90^\circ - \frac{\alpha}{2},$$

that is $\angle BAD = \angle BDA$ and so $\triangle ABD$ is an isosceles triangle: $AB = BD = r$.

Draw a circle centered at B with radius r . Since $\angle ACD = \frac{\alpha}{2}$, point C lies on the circle (analyze this situation on your own). Consequently, $BC = r$. We have $AB = BC = AC$, i.e., $\triangle ABC$ is equilateral; $\angle ABC = 60^\circ$ and $\angle BDC = \angle BCD = 60^\circ + \frac{\alpha}{2}$. Hence, $\angle ADC = (90^\circ - \frac{\alpha}{2}) + (60^\circ + \frac{\alpha}{2}) = 150^\circ$.

16. A) Draw three mutually perpendicular planes through the center of the cube. They divide the cube into 8 smaller cubes with edge $\frac{1}{2}$. Since there are 9 points, two of the points can be found in at least one of the 8 cubes. The distance between the points does not exceed the length of the diagonal of a smaller cube, i.e., it does not exceed $\frac{\sqrt{3}}{2}$. Therefore, the two points are the desired ones.

B) Consider two subcases:

a) Draw three mutually perpendicular planes through the center of the cube. They divide the cube into 8 smaller cubes of edge 1. If some two of the 8 points are in one small cube, we are in the situation solved in A) and everything is proved.

Therefore, assume that each cube has exactly one of the 8 points. Suppose now that the distance between any two of the points is greater than 1.

Denote the distances from each of the points to the vertex of the small cube *nearest* to this point by $d_1 \geq d_2 \geq \dots \geq d_8$. Choose a smaller cube such that its point A corresponds to the distance d_1 . We can assume without loss of generality that A is within the black triangle of the upper face in the right-hand cube nearest to us, see Fig. 111 a).

Consider its neighboring non-black cube on Fig. 111 a) and let the point B on this cube correspond to the distance to the vertex V , which is equal to d_i .

Draw the ball of radius d_i centered at V and its intersections with the faces of this cube. We will produce three quarter-circles of radius d_i on the faces. It is on one of them that the point B lies. But then it is easy to prove (there are many ways of doing this, see below) that the distance from A to any point of an arbitrary quarter-circle centered at V does not exceed 1.

Here is one of such proofs. The most difficult case is the one when B is on the rear face (invisible to us) of the neighboring cube. (Analyze the two other cases on your own.)

We introduce coordinate axes at the upper face of the front cube (the one that contains point A) and at the rear face of the neighboring cube (the one that contains point B) as shown in Fig. 111 a). The coordinates (z, t) of A satisfy the condition

$$\frac{1}{2} \geq z \geq t \tag{1}$$

and the coordinates (x, y) of B satisfy the condition

$$x^2 + y^2 \leq z^2 + t^2. \tag{2}$$

The squared distance between A and B is

$$|AB|^2 = (1 - z)^2 + x^2 + (t - y)^2,$$

which follows from triangles $\triangle AMB$ and $\triangle MBH$, since $AM \perp MH$ and $BH \perp MH$. After simplification and taking (2) into account we get:

$$|AB|^2 = 1 - 2z + (z^2 + t^2) + (x^2 + y^2) - 2yt \leq 1 - 2z + 2(z^2 + t^2) - 2yt.$$

Since $z^2 + t^2 \leq 2z^2 \leq z \leq z + yt$ (we have made use of (1)), we have:

$$-2z + 2(z^2 + t^2) - 2yt \leq 2(z^2 + t^2 - (z + yt)) \leq 0$$

and, therefore, $|AB|^2 < 1$.

This contradiction proves the statement of the problem.

b) Let us place the cube on its vertex V so that one of its great diagonals is perpendicular to the horizontal plane. Put a point in each of the other vertices and number them 1 to 7, as shown in Fig. 111 b).

FIGURE 111. (Sol. A16)

Move points 1, 2, 3 over a short distance ε along the edges $1V$, $2V$ and $3V$; we get points $1'$, $2'$, $3'$. Then choose a number δ many times smaller than ε , e.g. 100 times smaller, and move points 4, 5, 6 to V over the distance δ along the diagonals of the squares shown in Fig. 111 a) and denote the new points by $4'$, $5'$, $6'$. Point 7 will not be moved.

It is easy to check that the distance between any two of 7 “hatched” (see *Prerequisites* on Dirichlet’s principle) points is strictly greater than 1 (verify it yourself).

17. a) If the planes are parallel, the statement of the problem is obvious and so it suffices to consider the case when they are not parallel. Project the solid body under consideration to the intersection line l of the given planes. We get segment AB .

On the other hand, the projection of the body to l can be obtained by projecting the body first to any of the given planes and then by projecting the projection thus obtained to l . The result is that both circles — projections of the body to our planes — are projected *onto* the segment AB whose length coincides, therefore, with the length of the diameter of each circle. Hence, the circles are identical.

REMARK. The body is not necessarily a ball: it can be of a complex shape, e.g. neither convex nor flat. This body lies in the intersection of two identical infinite cylinders perpendicular to the planes.

b) Just as in a), we can prove that the projections of the vertices of each polygon on l , the intersection line of the planes, are the same points A_1, \dots, A_n . In addition, the centers of both regular n -gons are projected into the same point O on l (to prove this use the theorem on three perpendiculars). Denote the radii of the circles circumscribed around the regular n -gons by R and R' ; it is obviously sufficient to prove that $R = R'$. So let us find the sum $OA_1^2 + \dots + OA_n^2$:

$$\sum_{k=1}^n OA_k^2 = \sum_{k=0}^{n-1} \left[R \cos \left(\alpha + \frac{2\pi k}{n} \right) \right]^2 = R^2 \sum_{k=0}^{n-1} \frac{\cos^2 \left(\alpha + \frac{2\pi k}{n} \right) + 1}{2} = \frac{nR^2}{2}.$$

Here we made use of the fact that

$$\sum_{k=0}^{n-1} \cos 2 \left(\alpha + \frac{2\pi k}{n} \right) = 0;$$

this equality follows from the fact that *the sum of vectors drawn from the center of a regular polygon to its vertices is zero*.

In the same way, $\sum_{k=1}^n OA_k^2 = \frac{nR'^2}{2}$; hence, $R = R'$ and the two n -gons are equal, Q.E.D.

19. Let us assume the contrary and let, for example,

$$\sqrt{2} = 1. \underbrace{\dots 77 \dots 7}_{k \quad 5000001} \dots, \quad k \leq 4999998.$$

Then $A = \frac{r}{10^k} + \frac{7}{9 \times 10^k}$, where r consists of the integer part and the first k digits of the decimal representation of $\sqrt{2}$, approximates $\sqrt{2}$ with an accuracy to $10^{-k-5000001}$ and $|A^2 - 2| = (A + \sqrt{2})|A - \sqrt{2}| \leq 3 \cdot 10^{-k-500001} < 10^{-k-5000000}$.

But on the other hand, $|A^2 - 2|$ is a rational number with denominator $(9 \cdot 10^k)^2$; hence, $|A^2 - 2| \geq \frac{1}{(9 \cdot 10^k)^2} > 10^{-2k-2}$. Therefore, $-k - 5000000 < -2k - 2$ or $k > 4999998$. Contradiction.

20. a) If n airplanes flew to the airfield O from the airfields A_1, \dots, A_n , this means that in each triangle $\triangle A_1OA_{i+1}$ the sides OA_i and OA_{i+1} are smaller than A_iA_{i+1} . Hence, $\angle A_iOA_{i+1} > 60^\circ$ for any $i = 1, \dots, n-1$.

But then the sum of these angles is greater than $n \times 60^\circ$ and, on the other hand, it is 360° , implying $n < 6$, i.e., $n \leq 5$.

b) For both the spherical and the flat triangles A_iOA_{i+1} it is also true that the greater angle subtends the longer side. Therefore the above reasoning can be applied literally to the case of the sphere. (The sum of the angles of a spherical triangle is $> 180^\circ$.)

22. Obviously, $\cos \alpha = \pm \frac{4}{5}$. Let for example, $\cos \alpha = \frac{4}{5}$.

Set $\sin k\alpha = \frac{a_k}{5^k}$, $\cos k\alpha = \frac{b_k}{5^k}$. Then the trigonometric formulas for the sine and cosine of the sum yield

$$a_{k+1} = 4a_k + 3b_k, \quad b_{k+1} = -3a_k + 4b_k.$$

This immediately implies that a_k and b_k are integer for all k . It remains to prove that they are not divisible by 5. The easiest way to do this is to notice (and prove by induction) that $a_k \equiv 3^k \not\equiv 0 \pmod{5}$ and $b_k \equiv 3^{k-1}4 \not\equiv 0 \pmod{5}$.

23. Note that if the three positions of the switch are labeled by 1, 2, 3 and the colors of the bulbs are also labeled by 1, 2, 3 then the situation described in the problem coincides with that of Problem 30.2.10.5 (its solution is given in Part 2).

25. Multiply the equation by $\cos\left(\frac{3\pi}{11}\right)$ and express $\sin\left(\frac{k\pi}{11}\right) \cdot \cos\left(\frac{3\pi}{11}\right)$ and $\cos^2\left(\frac{3\pi}{11}\right)$ as a sum. We get

$$\sin\left(\frac{3\pi}{11}\right) + 2\sin\left(\frac{5\pi}{11}\right) - 2\sin\left(\frac{\pi}{11}\right) = \sqrt{11}\cos\left(\frac{3\pi}{11}\right).$$

Squaring this equation and replacing again all products in terms of sums, and the squares in terms of the doubled angles we get after simplification

$$1 + 2\cos\left(\frac{2\pi}{11}\right) + 2\cos\left(\frac{4\pi}{11}\right) + 2\cos\left(\frac{6\pi}{11}\right) + 2\cos\left(\frac{8\pi}{11}\right) + 2\cos\left(\frac{10\pi}{11}\right) = 0.$$

which is obviously equivalent to the initial equation.

The latter identity can be also obtained geometrically. Namely, put the center of the right 11-gon in the origin, align the 11-gon symmetrically with respect to the x -axis and consider the projections of all vectors from the origin into the vertices to the x -axis, see Fig. 112.

The sum of the projections is clearly equal to the left hand side of the above equation. But since the sum of these vectors is obviously equal to $\vec{0}$, so is the sum of their projections, as required.

26. Let us rewrite the given equation in the form

$$520(x-1)(yzt+y+z) + 520(zt+1) = 57(yzt+y+z)$$

Now, it is clear that if $x > 1$, then the left hand side is greater than the right hand side and there are no solutions.

FIGURE 112. (Sol. A25)

Thus, $x = 1$, and the equation takes the form

$$57(yzt + y + z) = 520(zt + 1)$$

or

$$57z = (520 - 57y)(zt + 1).$$

Since $0 < z < zt + 1$, we should have $57 > 520 - 57y > 0$ which is only possible for $y = 9$. Substituting $y = 9$ we get $57z = 7(zt + 1)$.

Applying similar trick for the third time we finally get $\overline{xyzt} = 1978$. (The problem was suggested in 1978.)

ANOTHER SOLUTION. If you can notice that the equation can be rewritten in the form:

$$x + \frac{1}{y + \frac{1}{z + \frac{1}{t}}} = 1 + \frac{1}{9 + \frac{1}{7 + \frac{1}{8}}}$$

you immediately get

$$x = 1, \quad y = 9, \quad z = 7, \quad t = 8.$$

Indeed, if two numbers, e.g. continued fractions, are equal, then their integer and fractional parts are equal, too. This implies the uniqueness of expression of a number as a finite continued fraction.

29. A) The areas of triangles OAH , OAB and OBM cannot be equal. Indeed, if we assume the opposite and the areas of the triangles OAH , OAB and OBM are equal, see Fig. 113 a), then the fact that $S_{OAB} = S_{OBM}$ implies that $OA = OM$ (the heights of these triangles, dropped from the common vertex B to the bases coincide) and the fact that $S_{OAB} = S_{OAM}$ implies that $OB = OH$. Thus, BH and BM are divided in halves at O . Hence, $ABMH$ is a parallelogram and so BM and AH are parallel. This contradicts the fact that they meet at C .

Consequently, if some three areas of four considered are the same and are equal to S , then one of the parts with this area is quadrilateral $HOMC$. The following cases are possible:

B) See Fig. 113 b). In this case the second solution (given below) is even simpler than arguments in case C) and the answer is $S = \frac{1}{6}$.

C) Suppose that triangles AOB and BOM have the same areas S and the area of $\triangle AOH$ is $s = 1 - 3S$. Connect H and M with a segment, see Fig. 113 b).

Let $BC = a$, $BM = x$, $AC = b$, $CH = y$. The ratio $\frac{BM}{MC}$ is equal to the ratio of the areas of the triangles with the bases BM and MC and a common vertex A , i.e., $\frac{x}{a-x} = \frac{2S}{S+s}$.

Similarly, having considered $\triangle ABH$ and $\triangle CBH$ we get $\frac{y}{b-y} = \frac{2S}{S+s}$. Consequently, $\frac{x}{a-x} = \frac{y}{b-y}$, whence

$$x(b-y) = y(a-x) \iff bx = ay \iff y = \frac{xb}{a}.$$

Denote: $|AB| = c$. From $\triangle ABM$ we have

$$S = \frac{1}{2} \cdot S_{ABM} = \frac{1}{2} \cdot \frac{1}{2} \cdot xc \sin B = \frac{1}{4} bx \sin C$$

because $c \sin B = b \sin C$ by the law of sines.

To define the areas of all four parts into which $\triangle ABC$ is divided, we can calculate s by two methods.

On the one hand, s is the area of $\triangle AHO$:

$$s = 1 - 3S = 1 - \frac{3}{2}S_{\triangle ABM} = \frac{1}{2}ab \sin C - \frac{3}{4}bx \sin C = \frac{1}{2}b \left(a - \frac{3}{2}x \right) \sin C. \quad (1)$$

On the other hand, $S - s$ is the area of $\triangle HMC$, i.e., it is equal to $\frac{1}{2}y(a-x)\sin C$, whence $s = S - \frac{1}{2}y(a-x)\sin C$. By substituting $S = \frac{1}{4}bx \sin C$ and $y = \frac{b}{a}x$ into (1) we get

$$s = \frac{1}{4}bx \sin C - \frac{1}{2}\frac{b}{a}x(a-x)\sin C = \frac{1}{2}bx \left(\frac{1}{2} - \frac{1}{a}(a-x) \right) \sin C. \quad (2)$$

We equate the right-hand sides of (1) and (2):

$$\frac{1}{2}b \left(a - \frac{3}{2} \right) \sin C = \frac{1}{2}bx \left(\frac{1}{2} - \frac{1}{a}(a-x) \right) \sin C \iff \left(a - \frac{3}{2} \right) = x \left(\frac{x}{a} - \frac{1}{2} \right)$$

therefrom we get the following quadratic equation for x :

$$x^2 + ax - a^2 = 0.$$

Hence,

$$x = a \frac{\sqrt{5}-1}{2} = \tau a, \quad y = \frac{xb}{a} = b \frac{\sqrt{5}-1}{2} = \tau b,$$

where $\tau = \frac{\sqrt{5}-1}{2}$. It remains to calculate S and s :

$$S = \frac{1}{4}bx \sin C = \frac{1}{2}ab \sin C \frac{\sqrt{5}-1}{4} = \frac{\sqrt{5}-1}{4} = \frac{\tau}{2}$$

(since $\frac{1}{2}ab \sin C = 1$ by the hypothesis);

$$s = 1 - 3S = 1 - \frac{3}{2}\tau = \frac{7-3\sqrt{5}}{4}.$$

FIGURE 113. (Sol. A29)

ANOTHER SOLUTION. The same problem has a simpler solution if we use allow to use *affine transformations* of the plane — a composition of (a) a parallel translation and (b) a rotation in space with subsequent projection to original plane and (c) a homothety. Such a transformation does not change the ratio of areas of any two figures. Let us make use of the affine transformation which turns the original triangle into an equilateral one and solve the problem set for the triangle obtained.

Despite of the fact that the solution seems to satisfy only the particular case of an equilateral triangle, it nevertheless is a solution for all triangles because of the affine nature of the problem.

The solution is based on the same reasoning as above but is much simpler. Indeed, case b) becomes completely obvious and in case c)

$$x = y, \quad \frac{1}{2}(a-x)\frac{\sqrt{3}}{2} = S - s, \quad s + 3S = \frac{a\sqrt{3}}{4}, \quad \frac{1}{2}\frac{\sqrt{3}}{2}a = 2S$$

(a is the length of the side of the regular triangle) therefrom it is easy to get the same quadratic equation $x^2 + ax - a^2 = 0$ with the positive root $x = \tau a$.

The number $\tau = \frac{\sqrt{5}-1}{2}$, the positive root of the equation $x^2 + x - 1 = 0$, is a remarkable number in Mathematics and even has a special name: the “golden section” or “Mister Tau.” It has many interesting and beautiful properties, one of which is that its continued fraction expansion is one of the simplest possible:

$$\tau = \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}}$$

The golden section first appeared in geometry during the search for “golden” rectangles which remain similar to themselves after squares are cut off from them. Let the longer side of a golden rectangle be 1 and the shorter side be τ ; the “goldness” property is then $\frac{\tau}{1} = \frac{1-\tau}{\tau}$ equivalent to $\tau^2 + \tau - 1 = 0$.

Mr. τ appears in many other geometric problems, e.g., in a problem on a regular pentagon. Let $ABCDE$ be a regular pentagon, see Fig. 113 d). Then its side is τ times smaller than the diagonal ($AB = \tau BD$) and the diagonals divide each other in the ratio of $\tau : 1$ ($BT = \tau BD, DK = \tau AD$). You can work out a proof yourself or look up in [Cox].

But why did Mr. τ appear when we solved our problem of cutting a triangle? Could we anticipate it and perceive Mr. τ directly without calculations?

Since, as was mentioned above, the problem is of affine nature, it suffices to replace from the very beginning an arbitrary triangle with a special one for which everything is clear. We tried an equilateral triangle as a most natural simple example but it proved to be only a trifle simpler than the original one and we still had to calculate. So for this problem the obvious idea of simplicity does not fit. What we need is the triangle ABD composed of the diagonals BD, AD and the side AB of the regular pentagon $ABCD$ and the intercepts AT and BK in it, see Fig. 113 d).

Indeed, $S_{\triangle AOB} = S_{\triangle BOT}$ since $AO = OT$ (the diagonal AT of parallelogram $ABTE$ is divided in halves by the other diagonal BE). In addition, $S_{\triangle BOT} = S_{KOTD}$ because $\triangle BTO$ and $KOTD$ are equally composed, the former consisting of triangle BMH and rectangle $MHTO$ and the latter of respectively equal figures: triangle TRD and rectangle $OTRK$.

Finally, we get

$$S_{\triangle AOB} = S_{\triangle BOT} = S_{KOTD}.$$

The diagonals of a regular pentagon have a remarkable property:

$$BT = \tau BD \quad \text{and} \quad KD = \tau AD;$$

hence,

$$S_{\triangle AOB} = S_{\triangle BOT} = S_{KOTD}$$

should be true for the original triangle.

You may be interested to plot the straight lines AM and BH using a ruler and compass. For this it suffices to learn how to plot lengths τ having a scale unit.

It can be done, for example, as follows, see Fig. 113 e). Construct square $ABCD$ with side 1, find the midpoint K of AD and then draw the circle centered at K with radius KC using your compass. Then the intersection point C' of this circle with an extension of AD is such that $DC' = \tau$. Indeed,

$$KC = \sqrt{\frac{1}{4} + 1} = \frac{\sqrt{5}}{2} = KC', \quad DC' = KC' - KD = \frac{\sqrt{5}}{2} - \frac{1}{2} = \tau.$$

Now to plot a segment of length τa from the segment a it suffices to perform the construction shown on Fig. 113 b) and well-known from school textbooks.

Plotting the segments of lengths $x = \tau a$ and $y = \tau b$, plot points M and H on AC and BC , see Fig. 113 b), so that $BM = x, CH = y$ and then draw the required line segments AM and BH . The construction is completed.

30. If the i -th digit of the number $2 - \sqrt{2}$ is equal to a , then the i -th digit of the number $\sqrt{2} - 1$ is equal to $9 - a$. If $4\frac{1}{3} < a < 4\frac{2}{3}$, then $4\frac{1}{3} < 9 - a < 4\frac{2}{3}$. It remains to generalize this fact to arithmetic means.

31. Select two planets and draw the plane through them and the center of the Sun. Let this plane be the equator of the Sun. Then from the northern and southern poles of the Sun not more than 7 planets can be seen (the planets which belong to the equatorial plane are invisible) and from at least one of the poles ≤ 3 planets are seen. Therefore, the pole from which ≤ 3 planets are seen is the point desired.

33. It is easy to see that even the squares of size $\frac{1}{2}$ and $\frac{1}{3}$ cannot be squeezed inside any square whose side is less than $\frac{5}{6}$. In the square with side $\frac{5}{6}$ all squares might be put, since their total area is equal to $\frac{\pi^2}{6} - 1 < \left(\frac{5}{6}\right)^2$. (We used the fact that $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$; for its proof see any good text-book on Calculus.) This inequality certainly does not prove that they can actually be squeezed into a $\frac{5}{6} \times \frac{5}{6}$ square, one should guess an ingenious way to arrange them; look at Fig. 114. The bubbles on Fig. indicate the sized of the squares placed in the square indicated in the same way as the squares depicted on Fig. are placed in the given $\frac{5}{6} \times \frac{5}{6}$ squares.

FIGURE 114. (Sol. A33)

FIGURE 115. (Sol. A34)

34. The construction required is carried out as follows. Fix two distinct points, A and B , not diametrically opposite, on the surface of the ball. Draw two circles centered at A and B on the ball; let C and D be the intersection points of the circles. (It is important here that A and B are not the endpoints of the same diameter, because otherwise the circles drawn would have either merged or have an empty intersection.)

Then draw two more intersecting circles on the surface of the ball centered at C and D . Let M and N be their intersection points. It is easy to see that all four points — A, M, N, B — lie on the same great circle, see Fig. 115.

REMARKS. 1) To draw the great circle on a *sheet of paper*, construct, for example, triangle $\triangle AMB$ (or $\triangle AMN$, or $\triangle ANB$). This can be done by the standard method of transferring segments of size AM , AB and MB with the help of a compass to the plane (of the paper). The circle circumscribed around $\triangle AMB$ is the required one.

2) It is easy to draw this circle also on the ball itself, not just on the plane, if the legs of the compass can bend.

35. Let us numerate parts of both partitions 1 to 100: A_1, \dots, A_{100} and B_1, \dots, B_{100} . Let S_{ij} be the area of $A_i \cap B_j$. Make an array of size 100×100 , where the (i, j) -th number is S_{ij} . Since the parts A_i and B_j of the partitions are of the same area, we have:

$$\sum_{i=1}^n S_{ij} = \sum_{j=1}^n S_{ij} = \frac{1}{100}.$$

Define a “snake” to be a set of 100 numbers S_{ij} in distinct rows and distinct columns. If all numbers S_{ij} constituting the “snake” are nonzero, the required 100 points should be placed in the respective intersections $A_i \cap B_j$. Thus, it remains to prove the existence of a snake built of nonzero numbers; such a snake is briefly called a *nonzero snake*.

Observe (it is quite obvious) that for any k rows selected there are k columns in whose intersection points with the rows selected there are at least k nonzero numbers S_{ij} .

Indeed, if the sets A_{i_1}, \dots, A_{i_k} have a nonempty intersection with l elements B_{j_1}, \dots, B_{j_l} of the second system, then $A_{i_1} \cup \dots \cup A_{i_k} \subset B_{j_1} \cup \dots \cup B_{j_l}$. Hence, $k \leq l$. Contradiction.

Let us prove the existence of a nonzero snake by induction on size of a minor¹. Namely, let us prove that for any $k \times k$ minor with a nonzero snake and an arbitrary $(k + 1)$ -th row there exists the $(k + 1)$ -st column such that the $(k + 1) \times (k + 1)$ minor thus obtained also has a nonzero snake.

¹A $k \times k$ minor is a set of k^2 squares — the table formed by the points of intersection of k rows and k columns.

The base of the induction: Take any nonzero element (a 1×1 minor) and add any nonzero element from another row. Adding also two more elements which are in the same rows and columns we get a 2×2 minor with a nonzero snake.

The inductive step: Let the statement be true for all $k \times k$ minors. We can assume without loss of generality that this minor is in the top left-hand corner of the array. Add another row. Without loss of generality, let it be the $(k + 1)$ -st row. We will prove that it is possible to select a column such that the respective minor contains a nonzero snake.

By observation above, in one of the first $k + 1$ rows there exists a nonzero element A_{il} standing to the right of the minor, i.e., $i \leq k + 1$, $l > k$. If A_{il} stands in the $(k + 1)$ -st row, let just augment the minor with the l -th column and mark the element $A_{k+1,l}$ in it. We get a “nonzero snake” of size $(k + 1) \times (k + 1)$, see Fig. 116 a).

FIGURE 116. (Sol. A35)

If $i \leq k$, then the i -th row contains a marked element, A_{ij} with $j \leq k$. Let us delete from the minor the i -th row and the j -th column. We get a $(k - 1) \times (k - 1)$ minor with a marked nonzero snake. By the inductive hypothesis it can be augmented with the $(k + 1)$ -st row and a column to a $k \times k$ minor with a nonzero snake. Let us augment the minor again with the i -th row and the j -th column (with the element $A_{ij} \neq 0$ marked); if the j -th column is already added, let us augment the minor with the l -th column and mark $A_{il} \neq 0$. We get a $(k + 1) \times (k + 1)$ minor with a marked nonzero snake.

38. For example, one die is placed so that along its lateral sides stand 2, 3, 5, and 4, the other 39 dice being placed so that along their lateral sides stand 1, 2, 6, and 5. Denote the number of 1's, 2's, 6's, and 5's over the 2 of the first die by x , y , z , t , respectively. Let us try to find a position for which the sums of dots on the lateral sides were equal:

$$2 + 1 \cdot x + 2 \cdot y + 6 \cdot z + 5 \cdot t = 3 + 2 \cdot x + 6 \cdot y + 5 \cdot z + 1 \cdot t = 5 + 6 \cdot x + 5 \cdot y + 1 \cdot z + 2 \cdot t.$$

The first and second equalities yield, respectively:

$$(x - z) + 4(y - t) + 1 = 0, \quad 4(x - z) - (y - t) + 2 = 0,$$

therefrom we can determine $x - z$ and $y - t$ and see that they are not integer.

40. Let

$$a_k = k + \frac{1}{(k+1) + \dots + \frac{1}{n-1}} \quad \text{and} \quad b_k = k + \frac{1}{(k+1) + \dots + \frac{1}{(n-1) + \frac{1}{n}}}$$

be the “tails” of the given continued fraction beginning with the integer k and $h_k = |a_k - b_k|$. Then

$$a_{k-1} - b_{k-1} = \frac{1}{a_k} - \frac{1}{b_k} = \frac{b_k - a_k}{a_k b_k}, \quad k = 2, \dots, n-1,$$

so $h_{k-1} = \frac{h_k}{a_k b_k} < \frac{h_k}{k^2}$ (because $a_k > k$ and $b_k > k$) and $h_{n-1} = \frac{1}{n}$. Hence, the difference we are interested in is estimated as follows:

$$h_1 < \frac{h_2}{2^2} < \frac{h_3}{2^2 3^2} < \dots < \frac{h_{n-1}}{2^2 3^2 \dots (n-1)^2} = \frac{1}{(n-1)!n!}, \quad Q.E.D.$$

41. Let points A, B, C move along the circles centered at O_1, O_2, O_3 in the same direction at an angular speed ω ; let $A(t), B(t), C(t)$ be their positions at the moment of time t . We assume that the plane under consideration is a complex line and O is the origin. Then the centers of the circles are complex numbers c_1, c_2, c_3 , the original points are numbers z_1, z_2, z_3 , and the points $A(t), B(t), C(t)$ are numbers:

$$A(t) = c_1 + (z_1 - c_1)e^{i\omega t}; \quad B(t) = c_2 + (z_2 - c_2)e^{i\omega t}; \quad C(t) = c_3 + (z_3 - c_3)e^{i\omega t}.$$

The radii of the circles are $|z_1 - c_1|, |z_2 - c_2|$ and $|z_3 - c_3|$, respectively.

The center of mass of $\triangle A(t)B(t)C(t)$ is the complex number

$$Z(t) = \frac{1}{3}(A(t) + B(t) + C(t)) = \frac{1}{3}(c_1 + c_2 + c_3) + \left(\frac{1}{3}(z_1 + z_2 + z_3) - \frac{1}{3}(c_1 + c_2 + c_3) \right) e^{i\omega t}.$$

Consequently, the center of mass of $\triangle A(t)B(t)C(t)$ is moving at an angular speed ω in the same direction as the original points A, B, C over the circle centered at the point $\frac{1}{3}(c_1 + c_2 + c_3)$ and with radius $\frac{1}{3}|(z_1 + z_2 + z_3) - (c_1 + c_2 + c_3)|$. The center of this circle is the center of mass of $\triangle O_1O_2O_3$ and its radius R is the length of the segment connecting the centers of mass of $\triangle ABC$ and $\triangle O_1O_2O_3$.

43. It is easy to see from Fig. 117 that $\beta = \frac{\sphericalangle AnB}{2}$, $\gamma = \frac{\sphericalangle BmA}{2}$, therefrom $\beta + \gamma = \frac{1}{2}(\sphericalangle AnB + \sphericalangle BmA) = \frac{360^\circ}{2} = 180^\circ$, $\gamma = 180^\circ - \beta$. Hence, $\sin \gamma = \sin \beta$.

FIGURE 117. (Sol. A43)

By the law of sines, for $\triangle AMC$ we have

$$\frac{CA}{\sin \alpha} = \frac{CM}{\sin \beta} \implies \frac{CA}{CM} = \frac{\sin \alpha}{\sin \beta}$$

and for $\triangle BMD$

$$\frac{DB}{\sin \alpha} = \frac{DM}{\sin \gamma} \implies \frac{DB}{DM} = \frac{\sin \alpha}{\sin \gamma} = \frac{\sin \alpha}{\sin \beta}.$$

Hence, $\frac{CA}{CM} = \frac{DB}{DM}$, Q.E.D.

44. Obviously, we can assume that not all given numbers are divisible by p . On the other hand, among the given numbers there are two, a and b , that yield the same remainder when divided by p , i.e., $a - b = lp$.

Suppose that a and b are not divisible by p . Clearly, $d = (a, b)$, their GCD, is also the GCD of a and $a - b$. Then $\frac{a}{d} > \frac{a-b}{d} = p\frac{l}{d} \geq p$, as required.

Analyze on your own the case when a and b are divisible by p .

45. Denote the function $\sum_{n=1}^k a_n \cos nx$ by $P(x)$. We see that $P(x) \geq -1$ for all x under the condition of the problem.

Let us prove that $\sum_{l=0}^k P(\frac{2\pi l}{k+1}) = 0$. For this it suffices to prove that

$$\begin{aligned} \sum_{l=0}^k a_1 \cos\left(\frac{2\pi l}{k+1}\right) &= 0, \\ \sum_{l=0}^k a_2 \cos 2\left(\frac{2\pi l}{k+1}\right) &= 0, \\ \dots\dots\dots \\ \sum_{l=0}^k a_k \cos k\left(\frac{2\pi l}{k+1}\right) &= 0. \end{aligned}$$

But each of these sums is the sum of the real parts of complex numbers lying at the vertices of a regular polygon, i.e., it is actually equal to 0 and the required equality is proved.

It follows, therefore, that

$$P(0) = -\sum_{l=1}^k P\left(\frac{2\pi l}{k+1}\right)$$

and since $P\left(\frac{2\pi l}{k+1}\right) \geq -1$ for all $l = 1, \dots, k$, we have

$$P(0) \leq -\sum_{l=1}^k (-1) = k.$$

46. Clearly, the first prompt of Guesser is *arbitrary*. It divides 16 possible numbers into groups 1 + 4 + 6 + 4 + 1 (if the prompt is: “11111”, then the i -th group has numbers with i -many 1’s).

If we are unlucky as Guesser and got in the middle group we will be unable to guess.

Indeed, at the second prompt only the number of 1’s is important, not their order; 10001 divides 6 numbers into groups 3 + 3 and 10011 divides 6 numbers into groups 1 + 4 + 1. Therefore, it is clear that the third prompt can not divide a group of 4 (since no prompt divides into four groups; if we do not prompt a number of type 10011, we will be even unable to divide the group of 3).

REMARK. One can interpret the problem differently: since there stands a “five-digit number” in the formulation, we may think that it is assumed that the first number is necessarily a 1.

49. The simplest way to solve this problem is to regard the white plane as a complex line, and consider the locations of the man and the cat as complex numbers $z_m = z_m(t)$ and $z_c = z_c(t)$, respectively, that depend on time t (t is a real number). Assume that the cat is at point O at the moment $t = 0$.

The cat encircles the man in three stages.

Stage 1. Denote: $w = \frac{z_c}{z_m}$ (observe that $z_m \neq 0$: the man will not go to the origin *because the cat already was there*). At the first stage, the cat runs along a straight line until the absolute value of w becomes $|w| = 1 + \frac{\varepsilon}{2}$, where $\varepsilon = \lambda - 1 > 0$.

Stage 2. The cat moves in a special manner: so that $|w| = 1 + \frac{\varepsilon}{2}$ at all times. At the beginning of the second stage, $\arg z_c = 0$, $0 < \arg z_m < 2\pi$, $\arg w = -\arg z_m < 0$; all the angles (\arg ’s) depend continuously on t . If the cat wanted to make $w = \text{const}$, it would suffice for it to move at a speed of $1 + \frac{\varepsilon}{2}$. Therefore, it has an excess of speed $\frac{\varepsilon}{2}$ which it can use to change the argument of w . The cat runs over a distance not greater than $t(1 + \varepsilon)$ over the time interval t , so $|z_c| \leq t(1 + \varepsilon)$ and the cat can ensure that the rate of variation of $\arg w$ is at least

$$\frac{\frac{\varepsilon}{2}}{|z|} \geq \frac{\frac{\varepsilon}{2}}{t(1 + \varepsilon)}$$

Since $\int_c^\infty \frac{dt}{t} = \infty$, the variation of $\arg w$ can be infinitely large. At stage 2, therefore, the cat increases $\arg w$ at the maximum possible rate. The second stage terminates when $\arg w$ becomes positive.

Stage 3. Finally, the cat moves so that $|z_c|$ does not increase, $|w| \leq 1 + \frac{\varepsilon}{2}$ and $\arg z_c$ increases at a rate limited from below by a constant. But the man cannot run away and the path of the cat closes after a finite time.

50. Consider one of the given points and all annuli in which it is contained. The centers of these annuli constitute an annulus with the inner radius 2 and the outer one 3. The area of this annulus equals $9\pi - 4\pi = 5\pi$.

Let us construct 650 of such annuli for each given point; all of them are contained in the disc of radius 19. Suppose that the statement of the problem is false. Then no point of the plane is contained simultaneously in 10 of the annuli constructed. Therefore, the area of the union of the annuli is greater than $650 \cdot 5 \cdot \frac{\pi}{9} = \pi \cdot 361.11\dots$

But this is greater than the area of the disc of radius 19 in which, as we have already shown, all of the annuli are contained. Contradiction.

51. Let us prove by induction that for any number x and an integer n there exists $\varepsilon > 0$ such that no number from the segment $]x - \varepsilon, x[$ can be represented as the sum of n numbers, each inverse to a positive integer.

The *base of induction*, $n = 0$ is obvious: take $\varepsilon = x$. The *inductive step*: Let the statement be true for an n ; let us prove that it holds for $n + 1$.

For every integer $q \leq 2 \frac{n+1}{x}$ there exists by the inductive hypothesis a number ε_q such that no number from the segment $]x - \varepsilon_q - \frac{1}{q}, x - \frac{1}{q}[$ can be represented as the sum $\frac{1}{a_1} + \dots + \frac{1}{a_n}$ for positive integers a_i , $i = 1, 2, \dots, n$. Hence, no number from the segment $]x - \varepsilon_q, x[$ can be represented as $\frac{1}{a_1} + \dots + \frac{1}{a_n} + \frac{1}{q}$.

Let $\varepsilon = \min_{1 \leq q \leq 2 \frac{n+1}{x}} \varepsilon_q$. Then no number from the segment $]x - \varepsilon, x[$ can be represented as $\frac{1}{a_1} + \dots + \frac{1}{a_n} + \frac{1}{a_{n+1}}$, where $a_i \leq 2 \cdot \frac{n+1}{x}$ for $i = 1, 2, \dots, n+1$ (the a_i 's can be permuted and renumbered).

If $a_i > 2 \cdot \frac{n+1}{x}$ for all i , then $\frac{1}{a_1} + \dots + \frac{1}{a_n} + \frac{1}{a_{n+1}} < \frac{1}{2}x$ which for $\varepsilon < \frac{1}{2}x$ does not belong to the segment either. Q.E.D.

53. Each edge of the cube has a point of the polyhedron because otherwise the projection of the polyhedron along this edge would not coincide with the face. Take one point of the polyhedron on each edge of the cube and consider the new convex polyhedron with vertices at these points. Since the new polyhedron is a part of the original one, it suffices to prove that its volume is not less than one-third of the volume of the cube.

We can assume that the length of the edge of the cube is 1. The new polyhedron is produced by cutting off tetrahedrons from trihedral angles at the vertices of the cube. Let us prove that the sum of the volumes of two tetrahedrons corresponding to the vertices belonging to one edge of the cube does not exceed $\frac{1}{6}$. This sum is $\frac{1}{3}S_1h_1 + \frac{1}{3}S_2h_2$, where h_1 and h_2 are heights dropped to the opposite faces of the cube from the vertex of the polyhedron, which belongs to the given edge of the cube, and where S_1 and S_2 are areas of the respective faces of the tetrahedron. It remains to observe that $S_1 \leq \frac{1}{2}$, $S_2 < \frac{1}{2}$, and $h_1 + h_2 = 1$.

Four parallel edges of the cube set a partition of the cube's vertices into 4 pairs. Therefore, the volume of all cut-off tetrahedrons does not exceed $\frac{4}{6} = \frac{2}{3}$, i.e., the volume of the remaining part is $\geq \frac{1}{3}$. This estimate is a precise one: for cube $ABCD A_1 B_1 C_1 D_1$ the equality is attained for tetrahedrons $AB_1 C D_1$ and $A_1 B C_1 D$.

FIGURE 118. (Sol. A54)

FIGURE 119. (Sol. A55)

55. First, prove the following

Lemma: for an arbitrary triangle ABC and for arbitrary points B_1 on side AC and C_1 on AB the area of $\triangle CBM$ is greater than that of $\triangle C_1 B_1 M$, where M is the intersection point of BB_1 and CC_1 , see Fig. 119 a).

Now, observe that if $ABCDE$ is the original pentagon and $A_1 B_1 C_1 D_1 E_1$ is the smaller pentagon, then the number s calculated for the smaller pentagon is equal to

$$S_{A_1 B_1 C_1} + S_{B_1 C_1 D_1} + S_{C_1 D_1 E_1} + S_{D_1 E_1 A_1} - S_{A_1 B_1 C_1 D_1 E_1}.$$

(The same is true, of course, for the greater pentagon but we do not need this fact.) Thus, it is easy to see that the difference $S - s$ is equal to the sum of five differences: $S_{AA_1 B} - S_{A_1 B_1 E_1}$, etc. (see Fig. 119 b)) each of which is positive by Lemma. Q.E.D.

57. To construct a counterexample, suppose that at the first step two units stand aside; hence, at the second step the number between them will also be a 1. Let us depict this as

$$\begin{array}{c} 1 \quad 1 \\ 1 \end{array}$$

Further, suppose that at 3-rd step under the lowest 1 two 2's will be obtained and, therefore, at 4-th step we get

$$\begin{array}{c} 1 \quad 1 \\ 1 \\ 2 \quad 2 \\ 2 \end{array}$$

Such an arrangement can be obtained from the triangle

$$\begin{array}{cccc} 5 & 1 & 1 & 5 \\ & 3 & 1 & 3 \\ & & 2 & 2 \\ & & & 2 \end{array}$$

At the next step we can construct, e.g., the triangle

$$\begin{array}{cccccc} 13 & 5 & 1 & 1 & 5 & 13 \\ & 9 & 3 & 1 & 3 & 9 \\ & & 6 & 2 & 2 & 6 \\ & & & 4 & 2 & 4 \\ & & & & 3 & 3 \\ & & & & & 3 \end{array}$$

And so on. The method indicated makes it clear that it is always possible to obtain at the $2k$ -th step the number k under the initial 1. On the other hand, if at some moment all the numbers become equal, they remain equal to the same constant later, while our construction puts at the center the number k at the $2k$ -th step, i.e., the middle number grows monotonously. This is the counterexample desired.

ANOTHER DESCRIPTION OF THE SAME SOLUTION. Let the numbers in the first row be given by the formula $a_n = 2n^2 - 2n + 1$ for $n \in \mathbb{Z}$; therefore, the first row is of the form

$$\dots \quad 25 \quad 13 \quad 5 \quad 1 \quad 1 \quad 5 \quad 13 \quad 25 \quad \dots$$

Make sure on your own that after the first step all the numbers remain integers and after the second one each number accrues by 1. This easily implies that (a) the numbers will always remain positive integers, (b) they will never become equal.

59. Note that the total number of distinct spheres is finite: there are finitely many given points and, therefore, there are finitely many various centers of mass of distinct subsets of our points. Therefore, we immediately deduce that after a while we have a cycle of spheres. A sphere will eventually get another number and then the spheres will start to be counted cyclicly.

It remains to prove that the length of this cycle is equal to 1.

Lemma. The function $\sum_{i=1}^n |x - x_i|^2$ in the k -dimensional space attains its minimum if and only if x is the center of mass of points x_1, \dots, x_n .

PROOF. Indeed, let O be the center of mass of the points x_i , i.e. $\sum_{i=1}^n \vec{x}_i = 0$ (recall that the points are of equal mass). Then

$$S = \sum_{i=1}^n |x - x_i|^2 = \sum_{i=1}^n \sum_{j=1}^k (x^{(j)} - x_i^{(j)})^2 = \sum_{i=1}^n \sum_{j=1}^k \left((x^{(j)})^2 + (x_i^{(j)})^2 - 2x^{(j)}x_i^{(j)} \right).$$

Since $\sum_{i=1}^n \vec{x}_i = 0$, we deduce that $\sum_{i=1}^n x_i^{(j)} = 0$; hence, $\sum_{j=1}^k \sum_{i=1}^n x_i^{(j)} = 0$ and $S \geq \sum_{j=1}^k \sum_{i=1}^n (x_i^{(j)})^2$, where the equality occurs only in case $x^{(1)} = \dots = x^{(k)} = 0$, as required.

Now, consider the function

$$S(X) = \sum_{i: |XA_i| \leq 1} (1 - |XA_i|^2),$$

where the summation runs over the points A_i from our set lying inside the unit sphere centered at X . Let $X' \neq X$ be the center of mass of these points. Let us show that $S(X') > S(X)$.

Indeed, if under the shift of the center of the sphere from X to X' the content of points A_i does not vary, the inequality follows from Lemma. Adding new points causes addition of new positive summands to $S(X')$, while the elimination of old points causes deletion of negative summands. Therefore, a change of the content of points increases the value of $S(X')$. Hence, the value of $S(X)$ increases under each change of the center of mass; this eliminates cycles of length > 1 .

64. EXTENSION. Moreover, they(who??) can not be placed so that the *squares of distances* between the vertices were rational.

First, let us prove that *a triangle with vertices at nodes of the square lattice can not be equilateral*.

Assume the contrary; let there exist such a triangle and let a be the length of its side. Assume that the grid of the lattice is integer; then a^2 is an integer and the area of the triangle $S = \frac{\sqrt{3}}{4}a^2$ is an irrational number. But the area of the triangle with vertices at nodes of the lattice is a multiple of $\frac{1}{2}$ (see). Contradiction, Q.E.D.

Let us return to the solution of the problem. Let such a position exists. By zooming the whole picture the common denominator of all distances (or of all squares of distances if we are solving the extension of the problem) times we make all distances into integers. Choose a coordinate system so that the coordinates of the vertices of the square were $(c, 0)$, $(0, c)$, $(-c, 0)$, $(0, -c)$; let (x, y) be one of the vertices of the triangle. Since the distances from (x, y) to both $(c, 0)$ and $(-c, 0)$ are integer, it follows that $(x - c)^2 + y^2$ and $(x + c)^2 + y^2$ are integers; hence, so is their difference, $4cx$. Similarly, $4cy$ is an integer. Therefore, all vertices of the triangle should be situated at nodes of the square lattice with mesh of $\frac{1}{4c}$. But we have already proved that this is impossible. Contradiction.

65. The routes are designed as follows: we shall *always* turn right on the boundary of the town; while inside the town we shall only do so at the *marked* turning points. Until no turning points are marked, the tentative route is divided into $2n - 3$ rings (for $n \geq 2$); namely, one route around the town along the penultimate roads, $n - 2$ vertical and $n - 2$ horizontal ovals, see Fig. 120 a). If 4 rings meet at an intersection, we will unite the rings into one ring by marking the intersection point (with a "Turn Right" poster). In this way, by marking the intersection with coordinates $(2, 1)$ we will adjoin to the first ring two horizontal rings and a vertical one. Next, marking the intersection with coordinates $(3, 3)$ we will adjoin to the previous route a horizontal ring and two vertical ones, etc.

This method works if $n - 2$ is divisible by 3. In other cases we have to select several turning points and then proceed as above. The following turning points should be marked:

$$\begin{aligned} n = 3k + 2 &\geq 2: (2, 1), (3, 3), (5, 4), (6, 6), \dots (n - 2, n - 2) \\ n = 3k + 1 &\geq 4: (2, 1), (1, 2), (4, 3), (5, 5), (7, 6), \dots (n - 2, n - 2) \\ n = 3k &\geq 6: (2, 1), (1, 2), (3, 2), (4, 4), (6, 5), (7, 7), \dots (n - 2, n - 2) \end{aligned}$$

A singular case: $n = 3$. A case-by-case verification shows that in this case there exists no closed routes; but it is not difficult to construct a route with the beginning point and endpoint on the opposite sides of one road; see Fig. 120 b).

FIGURE 120. (Sol. A65)

FIGURE 121. (Sol. A66)

66. If the midpoints of the parallel sides in an octagon are connected with straight lines, these lines pass through the center O of the circle. Introduce the following notations, see Fig. 121:

$$\begin{aligned} x &= \frac{1}{2} \cup A_1 A_2, & z &= \frac{1}{2} \cup A_2 A_3, & v &= \frac{1}{2} \cup A_3 A_4, \\ y &= \frac{1}{2} \cup A_5 A_6, & t &= \frac{1}{2} \cup A_6 A_7, & w &= \frac{1}{2} \cup A_7 A_8. \end{aligned}$$

Since $x + z = t + y$ and $z + v = t + w$ (two pairs of vertical angles as shown on Fig. 121), it follows that $x - v = y - w$ and $x + w = y + v$. Therefore,

$$t + 2w + 2x + z = t + 2y + 2v + z,$$

i.e., line l connecting the midpoints of $\cup A_2 A_3$ and $\cup A_6 A_7$ cuts off the same sum of the arcs on the half-circles.

Since $\cup A_1 A_8$ and $\cup A_4 A_5$ supplement these sums to half-circles, $\cup A_1 A_8 = \cup A_4 A_5$ implying $A_1 A_8 = A_4 A_5$, Q.E.D.

69. a) For example, take for such numbers all n -digit numbers of the form $11\dots 11$, where n is a power of 3.

b) Let $S(N) = 2^n - 1$ and let the number consisting of n last digits of N be divisible by 2^n ; then N is also divisible by 2^n . For example, $N = 92112$ is divisible by 16.

71. Let $f(x) \neq 0$. Then

$$f(x - 2y) + f(x) = 2f(x - y)g(y), \quad f(x) + f(x + 2y) = 2f(x + y)g(y)$$

implying

$$f(x - 2y) + f(x + 2y) = -2f(x) + 2g(y)(f(x - y) + f(x + y)) = f(x)(-2 + 4g(y)^2).$$

On the other hand,

$$f(x - 2y) + f(x + 2y) = 2f(x)g(2y).$$

Dividing by $f(x)$ we get $g(2y) = -1 + 2g(y)^2 > -1$. Since $2y$ is an arbitrary number, we are done.

REMARK. For example, the functions $f(x) = \sin x$, $g(y) = \cos y$ possess the property. Accordingly, we have $\cos y > -1$.

73. First, suppose that all roots of the polynomial P can be divided into pairs of complex-conjugate non-real numbers, z_i and \bar{z}_i , as follows:

$$P(x) = [(x - z_1) \dots (x - z_n)][(x - \bar{z}_1) \dots (x - \bar{z}_n)] = P_1(x)P_2(x),$$

where $P_1(x) = (x - z_1) \dots (x - z_n)$; $P_2(x) = (x - \bar{z}_1) \dots (x - \bar{z}_n)$.

Then $P_2(x) = \overline{P_1(x)}$. Denote: $P_1(x) = Q(x) + iR(x)$, $P_2(x) = Q(x) - iR(x)$, where Q and R are polynomials with real coefficients. We get

$$P(x) = P_1(x)P_2(x) = Q(x)^2 + R(x)^2.$$

Now, let us turn to the general case and remember that any real polynomial can be expressed in the form

$$P(x) = a(x - x_1)^{k_1} \dots (x - x_s)^{k_s} \tilde{P}(x),$$

where $a \neq 0$ is the coefficient of the highest term, x_1, \dots, x_k all distinct real roots of $P(x)$, $\tilde{P}(x)$ a polynomial without real roots, i.e., $\tilde{P}(x)$ is the product of quadratic polynomials $(x - z)(x - \bar{z}) = x^2 + px + q$. If $P(x) \geq 0$ for some $x \geq 0$ (e.g. for $x > x_j$, $1 \leq j \leq s$) then $a > 0$.

Then, $a = b^2$, $b > 0$ and by the already proven $\tilde{P}(x) = \tilde{Q}(x)^2 + \tilde{R}(x)^2$. Suppose all k_j are even. Then $P(x) = b^2 Q_1^2(x) \tilde{P}(x) = (b Q_1 \tilde{Q})(x)^2 + (b Q_1 \tilde{R})(x)^2$ and we are done. Now it suffices to prove that all k_j are indeed even.

It remains, moreover, to consider only one case: $P(x) = (x - x_1) \dots (x - x_s)$ with distinct roots. To conclude the proof show that $P(x_0) < 0$ for a real x_0 .

74. Let x be not divisible by 3. Then $x^2 \equiv 1 \pmod{3}$ implying $2y^2 \equiv 1 \pmod{3}$. But this means that $y^2 \equiv 2 \pmod{3}$ which is impossible. Therefore, x is divisible by 3, hence so is y . But then the left hand side is divisible by 9 and dividing by 3 we see that $2t^2 - z^2$ is divisible by 3. Hence, both t and z should be divisible by 3 by the same reasons. Thus, any solution (x, y, z, t) has a common divisor 3. This easily implies that the only solution is

$$x = y = z = t = 0.$$

75. a) Denote the position of the airplane at the moment t by $X(t)$ and let the whole time of the flight be T sec. Then $X(0) = \Gamma_1$, $X(T) = \Gamma_2$, where $0 \leq t \leq T$, $T > 1$.

Denote:

$$\alpha(t) = \angle X(t)AX(t+1), \quad \beta(t) = \angle X(t)BX(t+1).$$

Since the path of the plane is continuous, so are the functions $\alpha(t)$ and $\beta(t)$ defined for $0 \leq t \leq T-1$.

Let us draw a plane through $\Gamma_1\Gamma_2$ and $X(1)$. From Fig. 122 a) we see that $\alpha(0) > \beta(0)$ because $\alpha(0)$ is an exterior angle and $\beta(0)$ is an interior angle of $\triangle AX(1)B$.

Similarly, by drawing the plane through $\Gamma_1\Gamma_2$ and $X(T-1)$, we deduce from $\triangle AX(1)B$ that $\alpha(T-1) < \beta(T-1)$. Then the continuity of $\alpha(t)$ and $\beta(t)$ implies the existence of a moment t_0 such that $\alpha(t_0) = \beta(t_0)$, Q.E.D.

b) Fig. 122 b) shows a half-circle of diameter Γ_1B (the whole picture is two-dimensional); Γ_2 lies on the half-circle near B and A belongs to the diameter near Γ_1 . Then $\beta = \frac{1}{2}\cup XY$ and $\alpha > \frac{1}{2}\cup XY$, since $\alpha = \frac{1}{2}\cup XY + \cup X'Y'$, where X' and Y' are the intersection points of XA and YA , respectively, with the second part of the circle. Therefore, $\alpha(t) > \beta(t)$ for all t .

FIGURE 122. (Sol. A75)

76. Let us assume that the shades are *closed*, i.e., the boundary of a shade is a part of the shade. (If the shades are open the same proof is applicable with slight modifications.)

Assume that the statement of the problem is false and quadrilateral $ABCD$ with discs K_A, K_B, K_C, K_D is a counterexample. We may assume that the discs are in *general position*, i.e., are not tangent to each other, no 3 of them intersect at one point and the pair-wise intersection points do not lie on the sides and diagonals of the quadrilateral; otherwise — i.e., for a *singular* position — we can slightly enlarge the discs without violating the nature of the counterexample.

Let us prove that the system of discs indicated should possess the following properties:

1°. *No three discs have a common point.* Indeed, if, say, K_A, K_B and K_C have a common point, they cover $\triangle ABC$.

2°. *The points of pair-wise intersection of circles (the boundaries of discs K_A, \dots, K_D) cannot lie inside $ABCD$.* Indeed, if such a point would have existed, it would have been covered by one more disc (in order to exclude the possibility of a lighted spot in its neighborhood). This contradicts property 1°.

3°. *None of the discs can intersect two other discs.* Indeed, if, say, K_A intersects K_B and K_C , then the lighted (after K_D is deleted) part of $\triangle ABC$ can not intersect segments AB and AC and can not have vertices inside $\triangle ABC$ (by property 2°). It is not difficult to observe that this is impossible.

The discs with centers at the vertices of the quadrilateral possessing property 3° can not, however, cover the whole quadrilateral. The contradiction obtained proves the statement of the problem.

77. On the line, introduce a coordinate system so that at the Beginning of Time, simply called initial moment, one bacterium, call it F for first, were in the origin. It survives at the next moment only if there is a bacterium in one of the points: (with coordinates) -1 or $\sqrt{2}$. These bacteria, in their turn, survive only if at the initial moment there were bacteria at points $-2, -1 + \sqrt{2}$ or $2\sqrt{2}$. And so on.

Consider all the bacteria that at the initial momets occupied points with coordinates $-n + m\sqrt{2}$ for positive integer n, m . Since the total number of all bacteria is finite, there finitely many such chosen bacteria. So there is one bacterium with the maximal m . If there are several such species, take the left-most of them, corresponding to the greatest value of n ; call it B .

It is clear now that there are no bacteria at points $-(n+1) + m\sqrt{2}$ and $-n + (m+1)\sqrt{2}$ and bacterium B is doomed to inevitable death.

Now it is not difficult to see that all the bacteria living at points with coordinates $-k + m\sqrt{2}$ for all k will eventually die out, too.

Now apply the inverse induction on m to see that in the end bacterium F will also perish. Since it was chosen at random (we could have shifted the origin), the statement is proved.

78. Let a k -configuration $C(k)$ be an arrangement of a finite number of points such that there exists at least k points at distance 1 from any of the points.

Take a k -configuration and move it a solid body by a vector of length 1 so that none of the points thus moved coincide with any of the initial points of the configuration. The new set of points, the union of the initial ones and the moved ones, clearly constitute a $(k+1)$ -configuration. The endpoints of a segment of length 1 is a 1-configuration; a 1000-configuration contains 2^{1000} points — the projections of the vertices of a 1000-dimensional unit cube onto a two-dimensional plane.

Let us try to find a configuration of points with the number of points $< 2^{1000}$. Denote by $A+B$ the vector sum of A and B — the set of all pairwise sums of points from A and B (each point being identified here with the endpoints of the vector equal to the sum of the two vectors corresponding to the summands).

Clearly, if A has m points and B has n points then $A+B$ has no more than $m \cdot n$ points.

Let $|C(k)|$ be the minimal possible number of points in a k -configuration $C(k)$. Let us prove that $|C(m+n)| \leq |C(m)| \cdot |C(n)|$.

Consider any point $a_i + b_j \in C(m) + C(n)$; using point a_i we can find all the points $a_{i_1}, \dots, a_{i_m} \in C(m)$ at distance 1 from a_i and using point b_j we can find all points $b_{j_1}, \dots, b_{j_n} \in C(n)$ at distance 1 from b_j . (We have to have all points $b_j + a_{i_k}$ distinct from any of the points $a_i + b_{j_l}$.) Then the distance between any of the points $b_j + a_{i_1}, \dots, b_j + a_{i_m}$ or any of the points $a_i + b_{j_1}, \dots, a_i + b_{j_n}$ is equal to 1:

$$|(b_j + a_{i_s}) - (b_j + a_{i_t})| = |a_{i_s} - a_{i_t}| = 1, \quad |(a_i + b_{j_s}) - (a_i + b_{j_t})| = |b_{j_s} - b_{j_t}| = 1.$$

But there is a total of $|C(m)| \cdot |C(n)|$ points in the set $C(m) + C(n)$ (since there are that many pairs (a_i, b_j)) and our statement is proved. Therefore, we have $|C(2)| = 3$ (a triangle) and:

$$\begin{aligned} |C(1000)| &\leq |C(2)| \cdot |C(998)| \leq |C(2)|^2 \cdot |C(996)| \\ &\leq |C(2)|^3 \cdot |C(994)| \leq \dots \leq |C(2)|^{500} = 3^{500} < 2^{1000}. \end{aligned}$$

EXTENSION. For a 1000-configuration in space the same argument yields $|C(3)| = 4$ (a tetrahedron) and $|C(1000)| \leq |C(3)|^{\frac{999}{3}} \cdot |C(1)| = 4^{333} \cdot 2$.

79. The problem is equivalent to rolling a regular pentagon $ABCDE$ over its sides. Let us prove that prints of one point (not necessarily the vertex) eventually produce an *everywhere dense set*, i.e., in any disc of any radius there will be a print.

First, we roll pentagon twice but so that its vertex A does not move. The point C will produce prints C^* and C^{**} . It is easy to see that C^{**} is made by C after rotation of the vector \vec{AC} through an angle of $2\pi - 2\frac{3\pi}{5} = \frac{4\pi}{5}$.

Denote the rotation through the angle of ϕ about A by R_A^ϕ . In these notations $R_A^{4\pi/5}(C) = C^{**}$. We can demonstrate that the superposition (successive performance) of the rotations $R_A^{4\pi/5}$ and $R_B^{-4\pi/5}$ in any order amounts to the parallel translation by a vector whose length may be taken to be equal to 1.

It is also easy to deduce that by rolling the pentagon one can produce both rotations through angles $\frac{k\pi}{5}$, $k \in \mathbb{Z}$, and parallel translations by vectors with the angles between them equal also to an integer multiple of $\frac{\pi}{5}$.

Let us consider four such translations by vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4$ with the angle between the neighboring vectors being $\frac{\pi}{5}$. We see that the vector $\vec{f}_2 = \vec{e}_1 + \vec{e}_3$ is collinear with \vec{e}_2 and is of length $2 \cos \frac{\pi}{5}$; the vector $\vec{f}_3 = \vec{e}_2 + \vec{e}_4$ is collinear with \vec{e}_3 and is of the same length. Since $2 \cos \frac{\pi}{5}$ is an irrational number, the set of prints produced on the straight lines by \vec{f}_2 and \vec{f}_3 is everywhere dense on the plane, Q.E.D.

REMARK. The required statement can also be proved with the help of the following lemma: *the square is the only regular polygon with all its vertices at the nodes of a square lattice*.

80. Put all boxes into one large empty box. When we put n boxes into an empty box, the number of empty boxes increases by $n-1$ and the number of filled ones by 1.

So, if $k+1$ boxes are filled (k boxes by hypothesis and 1 box added at the beginning), then $(k+1)(n-1)+1$ boxes are empty. We have a total of $(k+1)(n-1)+1+(k+1) = (k+1)n+1$ boxes and should remove the large box that we added.

81. By the condition $x^x - x^y = y^x - y^y$. Hence,

$$(y^{x-y} - 1)y^y = x^y(x^{x-y} - 1), \quad \frac{x^y}{y^y} = \frac{y^{x-y} - 1}{x^{x-y} - 1}. \quad (*)$$

Let $x > y > 1$. We have: $x^y > y^y$ and $y^{x-y} - 1 < x^{x-y} - 1$ and, therefore, $\frac{x^y}{y^y} > 1 > \frac{y^{x-y} - 1}{x^{x-y} - 1}$. Contradiction with (*). Consequently, $x = y$.

82. With the help of the formula for the product of sines the formula to be proved takes the form

$$\frac{abc}{6} \sqrt{2 \cos \alpha \cos \beta \cos \gamma - \cos^2 \alpha - \cos^2 \beta - \cos^2 \gamma + 1}.$$

Let us prove the latter formula. Let us take the coordinate system so that the vertex O of tetrahedron $OABC$ were at the origin, edge a along the Ox -axis, b on the plane Oxy ; see Fig. 123.

FIGURE 123. (Sol. A82)

Then the coordinates of A and B are $(a, 0, 0)$ and $(b \cos \gamma, b \sin \gamma, 0)$, respectively, and coordinates (x, y, z) of point C can be found from the equations

$$|\vec{c}|^2 = x^2 + y^2 + z^2 = c^2; \quad (\vec{a}, \vec{c}) = ax = ac \cos \beta; \quad (\vec{b}, \vec{c}) = b \cos \gamma \cdot x + b \sin \gamma \cdot y = bc \cos \alpha$$

implying

$$x = c \cos \beta; \quad y = \frac{1}{\sin \gamma} (c \cos \alpha - c \cos \beta \cos \gamma)$$

$$z = \sqrt{c^2 - x^2 - y^2} = c \sqrt{1 - \cos^2 \beta - \frac{1}{\sin^2 \gamma} (\cos^2 \alpha - 2 \cos \alpha \cos \beta \cos \gamma + \cos^2 \beta \cos^2 \gamma)}.$$

the volume of tetrahedron is equal to

$$\frac{1}{6} ab \sin \gamma \cdot z = \frac{1}{6} abc \sqrt{\sin^2 \gamma - \cos^2 \beta \sin^2 \gamma - \cos^2 \alpha + 2 \cos \alpha \cos \beta \cos \gamma - \cos^2 \beta \cos^2 \gamma}.$$

Now, it remains to replace $\sin^2 \gamma$ with $1 - \cos^2 \gamma$ and $-\cos^2 \beta \sin^2 \gamma - \cos^2 \beta \cos^2 \gamma$ with $-\cos^2 \beta (\sin^2 \gamma + \cos^2 \gamma) = -\cos^2 \beta$.

83. a) Consider an arbitrary section of the body. It is a disc D_0 centered at O . Draw line l through O at a right angle to the plane of the disc. Then draw an arbitrary plane P through l . The section of the body by P is a disc and so a segment of l intercepted by this section — denote it AB — is the diameter of this disc. Indeed, let CD be the segment along which two mutually perpendicular sections intersect. Since AB meets CD at point O , the midpoint of CD , and $AB \perp CD$, we are done.

Finally, drawing all possible sections of the body passing through l we see that all of them are discs with diameter AB . And since all these sections fill the entire body, the latter is a ball with diameter AB , Q.E.D.

b) We do not know a solution elementary enough to fit in this book.¹

¹Well, actually, we do not know a non-elementary one either.

84. 1°. Each key should be in ≥ 6 copies (otherwise in the absence of ≤ 5 keyholders the strongbox is impossible to open);

2°. For any 6 people there is a key which only these persons possess (to enable the others to open the strongbox); but this key, as any other one, is in ≥ 6 copies; hence, only these 6 persons are holders of this key.

85. Let the angle between hands of the clock be equal to α . For the clock to show some minutes past twelve the hour hand must form an angle of x with the ray passing from the center to 12 and so $0 < x < \frac{360^\circ}{12}$ and the minute hand must form an angle of $x + \alpha$ on one side and $12x$ on the other side with the same ray.

From the equation $12x = x + \alpha$ we find: $x = \frac{\alpha}{12}$.

86. The following lemma can be directly verified.

Lemma. If a, b, c, d are positive and $\frac{a}{b} < \frac{c}{d}$, then $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.

Now, we have $0 < \tan \alpha_1 < \tan \alpha_2 < \dots < \tan \alpha_n$ and so

$$\begin{aligned} \tan \alpha_1 &= \frac{\sin \alpha_1}{\sin \alpha_2} < \frac{\sin \alpha_1 + \sin \alpha_2}{\cos \alpha_1 + \cos \alpha_2} < \frac{\sin \alpha_2}{\cos \alpha_2} = \tan \alpha_2 < \tan \alpha_3 = \frac{\sin \alpha_3}{\cos \alpha_3} \\ \implies \frac{\sin \alpha_1 + \sin \alpha_2}{\cos \alpha_1 + \cos \alpha_2} &< \frac{\sin \alpha_1 + \sin \alpha_2 + \sin \alpha_3}{\cos \alpha_1 + \cos \alpha_2 + \cos \alpha_3} < \tan \alpha_3 < \tan \alpha_4 = \frac{\sin \alpha_4}{\cos \alpha_4} \\ \implies \frac{\sin \alpha_1 + \sin \alpha_2 + \sin \alpha_3}{\cos \alpha_1 + \cos \alpha_2 + \cos \alpha_3} &< \frac{\sin \alpha_1 + \sin \alpha_2 + \sin \alpha_3 + \sin \alpha_4}{\cos \alpha_1 + \cos \alpha_2 + \cos \alpha_3 + \cos \alpha_4} < \tan \alpha_4 < \tan \alpha_5, \text{ etc.} \end{aligned}$$

We get what we want by applying Lemma n times.

87. Let $S = x + y + z$. Subtract $\frac{1}{x}$ from this equation. We get $-\frac{y+z}{xS} = \frac{y+z}{yz}$. Hence, $y+z=0$ or $xS = -yz$.

If we assume the contrary to the desired, then $y+z \neq 0$ and, therefore, $xS = -yz$. Similarly, we have $yS = -xz$, and $zS = -yx$. Dividing $xS = -yz$ by $yS = -xz$ we get $\frac{x}{y} = \frac{y}{x}$. In a similar way we get $\frac{x}{z} = \frac{z}{x}$, and $\frac{y}{z} = \frac{z}{y}$. By the hypothesis, $y \neq -z \neq x \neq -y$, so we conclude that $x = y = z$ and $S = 3x$, $xS = 3x^2 = -x^2$. Hence, $x = 0 = y = z$. But the denominators are nonzero. Contradiction.

88. Observe that

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$$

and so:

$$\begin{aligned} 1 &= \frac{1}{2} + \frac{1}{2} \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{6} \right) = \frac{1}{2} + \frac{1}{2 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 6} \quad \text{for } n = 4, \\ 1 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = \frac{1}{2} + \frac{1}{4} + \frac{1}{4} \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{6} \right) = \frac{1}{2} + \frac{1}{4} + \frac{1}{4 \cdot 2} + \frac{1}{4 \cdot 3} + \frac{1}{4 \cdot 6} \quad \text{for } n = 5 \\ 1 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8 \cdot 2} + \frac{1}{8 \cdot 3} + \frac{1}{8 \cdot 6} \quad \text{for } n = 6 \end{aligned}$$

and so on.

EXTENSION. Can all denominators be distinct odd numbers?

It turns out they can for all odd $n \geq 9$. Try to prove this on your own. Start with two examples:

$$\begin{aligned} 1 &= \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{15} + \frac{1}{35} + \frac{1}{45} + \frac{1}{231} \quad \text{for } n = 9 \\ 1 &= \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{231} = \\ &\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{15} + \frac{1}{35} + \frac{1}{45} + \frac{1}{5 \cdot 77} + \frac{1}{9 \cdot 77} + \frac{1}{45 \cdot 77} \quad \text{for } n = 11 \end{aligned}$$

Further on, at each step we replace the least of the fractions $\frac{1}{3m}$ (with $m = 15 \cdot 77$ for $n = 11$) with the sum $\frac{1}{5m} + \frac{1}{9m} + \frac{1}{4m}$ by increasing the number of summands by 2 each time.

89. Let us place the table on an uneven floor first so that the tips A and C of its opposite legs are on the floor while the tips B and D are above the floor at the same distance from it (we will denote this distance by $h > 0$, $ABCD$ being a square). Now, press on the table so that A and C puncture the floor and stop at a distance $-h$ from it (i.e. at the distance h below the floor). Then B and D are on the floor.

We can obtain the same position of the table by rotating it about the center through 90° , points A and C being tangent to the floor at all times until they arrive at their final positions D and B , respectively. We see that the distance $h(t)$ from B and D to the floor (at time t) varies continuously in t from h to $-h$ as t

varies from 0° to 90° , and A and C belonging to the floor at all times. So all four points A, B, C, D belong to the floor for some t and at this moment $h(t) = 0$.

EXTENSION. (We do not know a solution of these more general problems.) Consider the problem for a *rectangle* $ABCD$. An even more general problem (suggested by S. Tabachnikov) is the one when $ABCD$ is a *quadrilateral inscribed into a circle* (the table must be inscribed so as not to shake, for example, on a spherical floor of a sufficiently large radius).

90. It is rather difficult to find x, y, z directly. Let us solve the system for a, b, c with x, y, z as parameters. We have a linear system in a, b, c and, since $x \neq 0, y \neq 0, z \neq 0$, it has a solution (perhaps, non-unique if the discriminant of the matrix, equal to $1 - \frac{1}{x^2} - \frac{1}{y^2} - \frac{1}{z^2}$, vanishes). But $a = xz, b = yz, c = xy$ is a solution of the system (with or without the help of linear algebra). Therefore, the system is compatible and there are no other solutions.

$$\text{So } xyz = \sqrt{abc}; \text{ hence, } x = \sqrt{\frac{ac}{b}}, y = \sqrt{\frac{bc}{a}}, z = \sqrt{\frac{ab}{c}}.$$

94. Denote the lengths of the tetrahedron's edges at the base by x, y, z and those of the edges coming out of the vertex by a, b, c . Let $a \geq \max(b, c)$.

If $a < b + c$, then two triangles can be constructed from the edges a, b, c and x, y, z .

Let now $a \geq b + c$. Since $a < x + c$ and $a < b + z$, it follows that $2a < x + z + b + c \leq x + z + a$ whence $a < x + z$ and two triangles are constructed from the edges a, x, z and b, c, y .

Finally, take an edge of maximal length (a) and two adjacent edges (b, c) or (x, z); the triangle with side a can be constructed in at least one of the cases; the remaining three edges now form a face of the tetrahedron.

95. Solutions different from that hinted at in Hint can be obtained with the help of canonical equations of the straight lines:

$$\frac{x - x_i}{a_i} = \frac{y - y_i}{b_i} = \frac{z - z_i}{c_i}, \quad i = 1, 2, 3.$$

96. It is easy to see that after the balloon is inflated each of the net's strings (laces) can be seen from the center of the balloon at the same angle as an edge of the standard $1 \times 1 \times 1$ cube is seen from its center. This angle is equal to the angle at the vertex of an isosceles triangle with base 1 and side $\frac{\sqrt{3}}{2}$ (half the diagonal of the cube): $\alpha = 2 \arcsin \frac{1}{\sqrt{3}}$.

Now, the radius R of the balloon can be found from the equation: $10 = R\alpha$, see Fig. 124; hence,

$$R = \frac{5}{\arcsin\left(\frac{1}{\sqrt{3}}\right)}.$$

FIGURE 124. (Sol. A96)

FIGURE 125. (Sol. A97)

97. Consider a mapping f that converts the map¹ K_0 with a larger scale into the map map K_1 with the smaller scale: each point representing some spot in K_0 (by considering only the part of the town depicted on K_0 we assume that $K_0 \supset K_1$) is placed upon the point representing the same spot in K_1 . Denote the image of K_1 under the same mapping by K_2 , see Fig. 125.

¹The assumption that the map of Moscow is continuous was quite unjustified before the time of "glasnost" and probably still is: large portions of Moscow were "classified", i.e., restricted for "security reasons" and quite a number of scientists were busy producing more or less plausibly distorted maps without "white spots". But, kidding aside, this assumption is also dubious mathematically if we take into account the *fractal* theory, see e.g., K. Falconer *Fractal Geometry*, Wiley, Chichester ea, 1990 and refs therein (esp. on p. xxii).

Further, set $K_n = f(K_{n-1})$, $n = 1, 2, \dots$. The rectangles $K_0, K_1, K_2, \dots, K_n, \dots$ have just one common point, x , since the sizes of the rectangles tend to zero.

The point x is precisely the point for piercing. Indeed, it follows from $x \in K_{n-1}$ that $f(x) \in K_n$. Thus the point $f(x)$ also belongs to all rectangles but there is only one such point and so $x = f(x)$.

REMARK. In general, the following theorem is true:

Any continuous map of a rectangle onto itself has a fixed point.

So the statement of the problem remains true even if one map is crumpled before being placed on top of the other map and pierced.

98. Since $2222^{5555} \equiv 3^{5555} \equiv (-4)^{5555} \equiv -4^{5555} \pmod{7}$ and $5555^{2222} \equiv 4^{2222} \pmod{7}$, we have

$$\begin{aligned} 2222^{5555} + 5555^{2222} &\equiv 4^{2222} - 4^{5555} \equiv 4^{2222}(1 - 4^{3333}) \\ &\equiv 4^{2222}(1 - 64^{1111}) \equiv 4^{2222}(1 - 1^{1111}) \equiv 0 \pmod{7}. \end{aligned}$$

Therefore, $2222^{5555} + 5555^{2222}$ is a multiple of 7.

ANOTHER SOLUTION. Note that $4^3 = 64 \equiv 1 \pmod{7}$. Hence

$$2222^{5555} \equiv 3^{5555} \equiv (-4)^{5555} \equiv (-4)^5 \equiv (-4)^2 \equiv -2 \pmod{7}.$$

Similarly,

$$5555^{2222} \equiv 4^{2222} \equiv 4^2 \equiv 2 \pmod{7},$$

and we are done.

FIGURE 126. (Sol. A99)

99. A sketch of the required construction with nine threads is shown in Fig. 126.

A convenient way of making it is to take 3 pencils.

If all threads have the same length l and the rods the same length d , then, for the construction to be rigid, it is necessary to have $d = l\sqrt{1 + \frac{2}{\sqrt{3}}}$.

The endpoints of the rods in our construction are vertices of two equilateral triangles arranged in parallel planes which are perpendicular to the straight line connecting the centers of triangles and rotated through an angle relative to one another. The rods themselves lie on straight lines which cross pairwise. The connection of the rods and threads is the same as for an octahedron.

It is extremely difficult to prove the sufficiency of the above condition.

REMARK. This connection was invented in the 1960s by an architect, B. Fuller. Many different designs of this type have appeared since then.

In this section we generously borrow from Introduction to [Le], [BY] and from [GT].

Historical remarks

H.1. On this book. The book contains problems of all mathematical Olympiads held in Moscow since the first one, in 1935. For the first time all of them are provided with solutions or at least hints and answers.

This book is not the first compendium of problems of Moscow Mathematical Olympiads. Selected problems of Moscow Olympiads 1–15 were collected in [SCY]; those from Olympiads 1–27 were compiled almost completely and published in Russian in [Le]. The compiler, A. A. Leman, and all who helped him, did a tremendous job to put through that edition. Written with care and with easy to understand solutions, the collection [Le] has become a rarity long ago. Both [SCY] and [Le], however, only contained solutions to selected problems.

A critical review of these collections revealed a number of omissions and errors which we corrected as best as we could¹. We generously borrowed from Introduction of [Le] and from the book [GT], where most of the problems from Olympiads 1–49 are supplied with at least an answer or a hint.

In this book, we offer the reader for the first time all (see footnote) problems of the first 27 Olympiads and all problems of the Olympiads after the 49-th. (Problems from Olympiad 50 can be found in Chinese [GT*]. The book was published in precopyright era without anybody's consent².) No complete (and correct) solutions of ALL problems had ever been published yet. A new generation of high school students will be able to get acquainted with a great number of interesting and beautiful ideas contained in more than 2 000 problems of the Moscow Mathematical Olympiads and learn the history of these happenings.

The problems for the Olympiads were put together and composed by many generations of graduate, undergraduate and post graduate students (mainly) from the Moscow University. The preparatory problems, those of the Olympiads themselves, and the problems told to school pupils at consultation sessions and lectures constitute a very valuable material for study; a sort of the mathematical folklore.

Enthusiastic university students pester undergraduates and professors offering their problems, fiercely criticizing others' problems and demanding to create more and more of new problems to the pool. Sometimes the discussions are very heated; sometimes a problem is discussed in whispers and the speakers look around like conspirators. It means that they discuss a problem that has a chance to be accepted for the Olympiad. More often than not a problem is so transformed during the discussion that the author can hardly recognize his creation. Thus, the preparatory problems and those of the Olympiad are mainly the result of a collective brain storm.

Unfortunately, this most valuable folklore is lost to a great extent and partly beyond recover. It is only with great difficulties that we managed to restore some of the problems of Moscow Olympiads and sometimes even complete original solutions that seemed to have been lost.

H.2. On necessity of Olympiads. In the mid-1930s many Soviet mathematicians pondered about the need for cooperation with the high school to bring up the next mathematical generation. The training of future mathematicians should begin in their childhood, the earlier the better. Nobody is surprised to see a ballet dancer or a musician starting their career at the age of 8 or 6 or earlier. The explanation is that it is impossible for a teenager to master all intricacies of the dancing art or of music, without specialized training when a child, to develop the ear and the feeling of rhythm, the flexibility of knuckles or agility of

¹Concerning completeness, this goal seem to be out of reach. Apart from Olympiads held during WW II, several problems of one more Olympiad seem to have sunk in Lethe. As I. M. Yaglom writes in *Problems, Problems, Problems. History and Contemporaneity*, Matematika v shkole (Mathematics in School), No. 5, 1989, pp. 143–148 : “ . . . I can not figure out why neither Leman nor Galperin and Tolpygo addressed the participants of that Olympiad . . . for instance, me.

The 2-nd set of Olympiad 4 contained 5 problems, not 4. Of the problems of the 1-st set I remember the one which I found the most difficult at that time. I hope that *you* will not find it difficult *now*, but bear in mind that it was the 1-st set:

In space (not on one plane) 4 points are given; how many planes equidistant from these points are there? I do not bet that in the formulation of this 49-years old problem the parenthetical restriction was explicit. Perhaps, it was required to consider separately the other case as well.”

Now Isaak Moiseevich Yaglom is dead. So nobody will, probably, provide us with the lacking problems.

²Which is a pity: the compilers would have supplied with corrections. For free.

fingers, etc. And each year lost in childhood might only be compensated later with many years of incredibly hard work.

It would be wrong to think that the situation is different in science, particularly in mathematics. Just like in dancing or music, the years lost in childhood are difficult to compensate later on. The trouble is that mathematics requires some agility of mind, the ability to think in abstract terms, and some logic culture which are impossible to pick up even by hard work at the university. To be sure, all qualities making up what is called “mathematical abilities” can develop at an ordinary school without any special training of a teenager. This is a spontaneous process for born mathematicians which has taken place in all countries and at all times. For example, **S. Ramanujan** (1887–1920), a very famous Indian mathematician had practically no mathematical education.

The mathematical talent, however, like that of a musician, manifests itself quite early, as a rule. Moreover, when developed properly, a mathematician can make his most important discoveries when still quite young. For example, **Evariste Galois** (1811–1832), a French mathematician, had time in his short life to create an algebraic theory remarkable in its depth, which greatly advanced the development of mathematics. **Carl F. Gauss** (1777–1855) published his classical studies of constructions with a ruler and compass when 19 years old; several years later he presented the book “*Disquisitiones arithmeticae*” which has few equals in the history of mathematical science.

Participation of the mathematicians in school education is extremely essential. This was the case before the Revolution; this work was reinitiated after the Revolution only in 30s; in Leningrad by CMA B. N. Deloné and Prof. V. A. Tartakovsky, and in Moscow by CMA L. G. Shnirelman, and Prof. L. A. Lusternik (later a CMA). The first mathematical Olympiad for schoolchildren in the USSR was held in Leningrad in the spring of 1934.

H.3. On Moscow Mathematical Olympiads. In the spring of 1935, the Board of the Moscow Mathematical Society, following the example of Leningrad, decided to organize the 1-st Moscow Mathematical Olympiad. The organizing committee included all professors of mathematics from the Moscow University and was headed by P. S. Alexandrov, the then President of the Moscow Mathematical Society. The purpose of the Olympiad was to find the most talented students, to attract attention of the young people at large to some of the most important problems and methods of modern mathematics, and to show the kids, least partly, what Soviet mathematicians are working on, what progress they have made and what challenges they have.

314 high school students participated in the Olympiad, 120 of them took part in Set 2 (the final). Three students were awarded first prizes, five got second prizes and, in addition, 44 kids were given honorary prizes. A place at the top of the Olympiad determined for many their future scientific career.

It is interesting how problems in Set 2 were selected. Three series of problems were offered and they were designated *A*, *B* and *C*. A. N. Kolmogorov told us that it was done by his initiative in order to enable students with different mathematical mentalities — geometrical, computational (algorithmic), or combinatorially logical — to develop. (Details see in [Ko]). It is according to these types of thinking that the series of problems were selected for the first Olympiad.

H.4. Mathematical circles. The success of the 1-st Moscow Mathematical Olympiads helped to restructure the relations of the researchers with schoolchildren. This eventually brought about the School Mathematical Circle attached to Moscow University. It was organized by L. A. Lusternik, L. G. Shnirelman and I. M. Gelfand (a member of the US National Academy, the French Academy, and almost all other Academies; at the dawn of perestrojka he was finally elected member of the USSR Academy of Sciences). The Circle had two types of activities: lectures on all kinds of subjects and meetings of their members. The lectures were attended first by dozens and then by hundreds of boys and girls from all over Moscow. Initially, the lectures were addressed to 8-th to 10-th graders; later (since 1940) the lectures were delivered also for 7-th and 8-th graders. The lectures set forth in a popular form serious mathematical results, including the latest scientific achievements.

The subjects of the lectures were quite diverse. Here are some examples of the lectures delivered at different times of the circle’s existence:

L. S. Pontryagin. *What is topology?*

A. G. Kurosh. *What is algebra?*

N. A. Glagolev. *Construction using only the ruler and without compass.*

A. A. Lyapunov. *Thinking machines.*

A. I. Markushevich. *Areas and logarithms.*

L. G. Shnirelman. *Multidimensional geometry.*

B. N. Deloné. *Derivation of the seven crystal systems.*

A. N. Kolmogorov. *The fundamental theorem of algebra.*

S. L. Sobolev. *What is mathematical physics?*
 I. R. Shafarevich. *Solutions of equations in radicals.*
 L. A. Lusternik. *Convex figures.*
 P. S. Alexandrov. *Transfinite numbers.*
 S. A. Yanovskaya. *What does it mean to solve a problem?*
 Ya. S. Dubnov. *Errors in geometric proofs.*
 I. A. Kibel. *Mathematical methods of weather forecasting.*
 V. V. Golubev. *Why does an airplane fly?*
 I. M. Gelfand. *Dirichlet's principle.*
 V. A. Efremovich. *Non-Euclidean geometry.*
 B. V. Gnedenko. *How science studies random phenomena.*
 N. K. Bari. *Arithmetics of the infinite.*
 G. E. Shilov. *About a derivative.*
 R. L. Dobrushin. *Mathematical methods of linguistics.*
 V. G. Boltyansky. *Continued fractions and the musical scale.*
 I. M. Yaglom. *How can we measure information?*
 O. A. Oleinik. *Helly's theorem.*

This list is far from complete, of course: many hundreds of lectures have been delivered for schoolchildren during the circle's existence.

We can say that the mathematical circle of Moscow University helped to revise considerably the term "elementary mathematics" (when it implies the body of mathematical knowledge that could be made fully understandable to school pupils).

Thus, for example, in his lecture "The fundamental theorem of algebra" made in 1937, Acad. A. N. Kolmogorov set forth an essentially full proof of the theorem on the existence of a complex root in any algebraic equation. This proof (called in the mathematical circle "The Lady with a lap dog" after a story by Chekhov) was published later in exactly the same form in the book "What Is Mathematics?" by R. Courant and H. Robbins, [CR].

The same year, L. G. Shnirelman in his lecture "The group theory and its application to solving 3rd order equations" brought up the group theory considerations, which actually go back to Galois, to obtain an explicit formula for a solution of 3rd order equations.

B. N. Deloné in his lecture "The geometry of continued fractions" delivered in 1947 not only proved a subtle theorem on the best rational approximations of irrational numbers but also described an elegant results obtained by Hurvitz about the irrationalities worst approximated by rational numbers.

Acad. S. I. Sobolev delivered in 1940 a lecture "What is mathematical physics?" and very skillfully brought the description (at the level understandable to teenagers) up to the 2-nd order partial differential equations indicating qualitative differences in the behavior of their solutions.

Sometimes the lectures were accompanied by problems to be solved at home or on the spot. I. M. Gelfand offered particularly numerous problems to his audience. The boys and girls who knew well his manner of lecturing often preferred to sit as far as possible from the speaker not to be called to the blackboard to solve a problem.

Ya. S. Dubnov's lectures were interestingly arranged. Sometimes he delivered a series of two lectures, the first lecture offering some problems whose solutions (partly found by the listeners) were discussed at the second one (in two weeks' time).

While delivering a lecture on deduction arithmetics and Boolean algebra, A. N. Kolmogorov drew a plot of an electric circuit shown in Fig. H1 to place two switches near the door and over a bed, each switch being able to turn on and off a lamp in the room irrespective of the position of the other switch. At the end of the first hour he challenged the listeners to find during the break the circuit which would enable one to switch the light on and off from n places in the room.

FIGURE 1. (Sol. AH1)

During preparations for the lectures, the lecturers often found new elegant proofs of well-known theorems, obtained new generalizations of some facts that they knew earlier, and even made small mathematical discoveries. Unfortunately, most part of this very valuable material is lost forever.

H.5. How to run mathematical circles (after Shklyarsky). Along with the lectures, there were regular meetings of sections of the circle. They were conducted generally by senior and post graduate students from "mekh-mat", i.e., the Department of Mechanics¹ and Mathematics of Moscow University

¹A more adequate translation would be "Mechanical engineering" instead of the conventional "Mechanics".

(except two sections in 1936 and 1937: the section of *geometric methods of the number theory* was headed by L. G. Shnirelman and the section of *qualitative geometry* was headed by A. N. Kolmogorov).

At first, the reports at the meetings were made by schoolchildren themselves; but soon it was found that this form of work was non-productive. The trouble was that most of the reports were of little interest and boring for all members of the circle (with an exception, perhaps, of the reporter himself). After all, it is not enough to understand everything what is said in the mathematical text given by the section head to make a good report.

A well-made report must arouse the interest of the audience and make the listeners think over and over what they have heard; it should contain a clear presentation of the problems to be discussed, the main ideas of the solution should be emphasized, the beautiful and original parts of proofs should be vividly depicted, and so on. Besides, a lecture can rarely be good if the lecturer knows the subject only within the limits of the lecture. Therefore, a teenager's report is usually far inferior to that of an experienced teacher.

The radical change in the work of the sections was associated with the name of David Shklyarsky, a talented mathematician and brilliant teacher, who headed the circles 1938 till 1941 while still a student. (D. O. Shklyarsky was killed in a guerrilla combat in 1942 at the age of 23 during WW II.)

Reports of the school pupils at the meetings were practically abolished. Instead, the head of a section delivered a short lecture that contained as a rule a complete description of a small mathematical theory. Then, the members described their solutions of the problems given at the previous meeting. Problems of varying complexities enabled Shklyarsky to involve virtually all section members into active work and by repeating a member's solution he had two objectives in mind: the audience understood better the solution exposed by an expert while the author of the solution had a lesson: how to lucidly present a mathematical proof.

This system has born a wonderful fruit: in 1938, at the 4-th Olympiad, members of Shklyarsky's section took away half of the prizes (12 out of 24), including all 4 first prizes! The results of the 4-th Olympiad astonished the heads of the other sections and next year practically all of them followed this example. Since then the form of the work of the circle found by D. O. Shklyarsky became predominant.

From the very beginning a tradition was established to issue annually a small collection of preparatory problems for the next Olympiad, which was given to the circle members and to all who came to the Olympiad.

H.6. Examples of programs and syllabus of specialized sections of the circle.

The Geometric Probabilities series

The problem on a meeting: *Two persons agreed to meet at a certain place, each has to come to the place between 10 and 11 o'clock and waits for the other one for exactly 15 minutes. What is the probability of the meeting?* The basic geometric idea is that the probability depends on the area or volume of the figure formed in the space of the events by points corresponding to favorable events.

The problem of constructing a triangle given three segments. (All kinds of varieties of the basic problem: *a stick is randomly cut into three pieces; what is the probability that these pieces can form a triangle?*)

The Buffon problem on throwing a needle for experimental determination of π . *Throwing a closed convex curve on a piece of paper ruled with parallel lines find the probability of the curve crossing a line.*

Barbier's theorem on the length of curves of constant width (as a corollary of the above or of Buffon's theorem).

The "area" (measure) of a set of straight lines crossing a given arc.

Crofton's theorem and basic ideas of integral geometry.

Geometric Maximum and Minimum Problems

The rectification method as applied to problems on inscribed polygons of minimum perimeter. (The typical problem is to find a point the sum of whose distances to the vertices of a triangle is minimal.)

An isoperimetric problem for n -gons ($n = 3, 4$ and the general case). Polygons of the greatest perimeter inscribed into a circle; escribed polygons of the least perimeter.

An isoperimetric problem for arbitrary lines. Steiner's four-hinge method and its critique. The problem whether there exists a solution of the minimum or the maximum problem.

Blaschke's theorem on the existence of a converging subsequence of convex figures. Substantiation of Steiner's method. Other examples of application of Blaschke's theorem.

Variational methods including the search for maximal and minimal figures. (The typical problem is to draw a straight line through a point inside an angle so as to cut off a triangle of the least area; solution of the problem using the method of geometric differentiation.)

The section of algebra

(Subtitled "Generalization of the notion of the number")

Natural numbers (or, as they are more often called in science, **positive integers**) were the main building blocks for further constructions. A quotation from L. Kronecker: "Natural numbers were created by God; the rest was done by humans."

The solvability of the equations $x + a = b$; subtraction. Generalization of the set of numbers in order to make subtraction always possible. The integers as material for sufficiently meaningful constructions; the theory of numbers. Examples of the number theory problems.

The solvability of the linear equations $ax + b = 0$; rational numbers. The number as a result of measurement; the number axis. The possibility of using only rational numbers in problems concerning measurements of geometric and physical values.

The solvability of quadratic equations. The insolvability of the quadratic equation $x^2 - 2 = 0$. The solvability of linear equations — the existence of points where the x -axis crosses the straight lines $ax + b = y$ (with rational coefficients); the absence (within the given stock of numbers) of the crossing point of the x -axis with the parabola $y = x^2 - 2$. Quadratic radicals. The solvability of all quadratic equations with real roots (the existence of the crossing points of the x -axis with the parabolas $y = ax^2 + bx + c$, where a, b, c are numbers from the given stock).

Construction of the point $x = \sqrt{2}$ (the diagonal of a unit square). Segments that can be constructed with a ruler and compass. The proof of insolvability of the problem on duplication of a cube — the parabola $y = x^3 - 2$ "slips through the known points of the x -axis". Further generalization of our stock of numbers; the real radicals of any order.

The problem of solution of cubic equations. The proof of the fact that the cubic parabola $y = x^3 - 4x - 2$ "slips through the points of the x -axis" (the irreducible case of a solution of a cubic equation.) The need to enlarge yet the stock of numbers.

The problem on solution of cubic equations again. **Complex numbers.** Geometric interpretations of complex numbers — the complex numbers as points in a plane; complex numbers as operators of rotational dilation. De Moivre's formula and problems on complex numbers.

The problem on solution of cubic equations once again. Cardano's formula. Another extension of the set of (real) numbers — combinations of complex radicals. The problem of solution of 4-th order equations.

The 5-th order equations. The proof that the parabola $y = x^5 - 4x - 2$ "slips through the known points of the x -axis." The need for a new extension of the set of numbers. All kinds of roots of all kinds of algebraic equations.

Another approach to "numbers" — infinite decimal fractions. A new extension of the set of numbers proving that the number

$$0.11000100000000000000000010000\dots (= \sum_{n \geq 1} 10^{-n!})$$

is not a root of any algebraic equation.

A different extension of the notion of numbers: quaternions as operators of rotational dilation in plane and in space. The geometric theory of quaternions. The quaternions and vectors; operation of vector algebra in space; problems. Frobenius' theorem.

Another extension of the notion of numbers — complex numbers, dual numbers ($\mathbb{Z}[x]/(x^2)$) and double numbers ($\mathbb{Z}[x]/(x^2 - 1)$). Geometric interpretation of dual numbers (as directed straight lines of a plane.) Geometric applications of double numbers. The idea of Hurvitz' theorem and of its generalizations.

(Of course, even in this extremely "theoretical" orientation of the circle, quite a few materials were left for exercises of the members on their own.)

H.7. Books for mathematical circles. In 1950, "Gostekhizdat" (The State Technical Publishers)¹ began publication of a special series of booklets under the heading "Popular Lectures on Mathematics", most of which were brought about as a specially prepared edition of lectures delivered at the mathematical circle of Moscow University. Some of the lectures were also published in the collection "Mathematical Education". A brief summary of some unpublished lectures is given in *Selected Lectures*. An extensive bibliography is given in [Le].

One of the main virtues of these books was, undoubtedly, their low cost.

H.8. How Moscow Mathematical Olympiads were arranged. The procedure has not practically changed since the 1-st Olympiad held in 1935.

The first 36 Olympiads (1935 through 1973) were held in two sets on Sundays at the end of March and early April (because of the kids' vacations).

Set 1 was selective, every participant being given 4 to 6 relatively easy problems (but with a catch!) and informed that it would be enough to solve two of the problems to get through to set 2.

A week after the first set, the problems were reviewed, different solutions and typical errors were pointed out, and the results were announced. Set 2 was held a week after that and all those who were successful in set 1 were invited (they made usually 30% to 50% of the total); sometimes teenagers who failed at set 1 were also allowed to participate in set 2 as well as those who missed set 1 for some reason² (generally, because of an illness; these were placed in special rooms.) The problems of the second set were considerably more difficult than those of set 1. In each set 5 or 6 hours were allotted to solve problems.

Finally, a week after, the problems of set 2 were discussed. Usually famous mathematicians were invited to review the problems conceived mostly by under and post-graduate students. The purpose was to combine descriptions of problem solutions with indications of broader perspectives in "big mathematics". Thus, one of us saw A. N. Kolmogorov in front of an audience that consisted mainly of several hundred of 8-th graders who started from two problems of the 38-th Olympiad (Problems

¹Later renamed "Fizmatgiz" (The Phys. and Math. State Publishers) and now called "Nauka" (Science). Whatever the name, the books were incredibly cheap *Où sont les naïges d'autrefois? Où sont elles...*

²This happened usually on recommendation of a member of the organizing committee who had noticed the originality of a teenager's thinking, and sometimes by personal request from a teenager; the objective was to *attract* kids to science, not just to gauge their sports skills.

38.8.3 and 38.8.4) and went over to discussion of modern issues of discrete mathematics (the graph theory and the information theory).

Then there was an awarding ceremony. The winners were given prizes: piles of mathematical books (1 m high) with dedicatory inscriptions. There were on the average about 10 first prizes (for different grades), twice that many second prizes and the thrice many third prizes. In addition, there were certificates of merit, of degrees 1 and 2. The results in set 1 were usually not taken into consideration in summing up the Olympiad and awarding the winners.

There was a three years' interval during the World War II from 1942 through 1944; during these years Moscow mathematicians held a number of olympiads in Ashkhabad and Kazan instead. Regrettably, we could not find any records of these events.

During the first five Olympiads all students were offered the same problems; beginning with Olympiad 6 the problems (and the students) were divided into two streams: for 7–8-th graders and for senior students.

Starting with the 15-th Olympiad (1952), the contest was held separately for each grade although some more interesting problems were given in parallel to several grades.

From the very beginning, a great assistance in organizing the Olympiads was rendered by the Moscow City Department of Public Education and the Lenin Advanced Training Institute for Teachers (abbreviated in Russian as MGPI, the Moscow Pedagogical Inst.). The staff of the latter together with experienced teachers and university mathematicians began to hold district olympiads in 1949. Their problems are given in ref. [SCY2]. They allowed to involve in mathematics a greater number of school pupils and not only senior graders but pupils of 5-th to 7-th grades as well.

While the mathematical circle for high school students had been the predominant form of extra-curricular mathematical activities for about a quarter of a century, the Moscow Olympiads focusing all lines of those activities, their forms have been noticeably diversified for the last 20 years. Specialized mathematical schools were set up and in 1963 many circles were combined in a new structure called “evening mathematical schools” to be followed a year later by “correspondence (extra-mural) mathematical schools”. Following the example of Moscow University, other higher educational establishments in Moscow started to hold mathematical olympiads of their own, and along with city olympiads there appeared a system of Republican, National, and, finally, International olympiads. However, the Moscow Olympiads continued to be something special for many years since their awards were considered to be a great honor and the standards of their problems were much higher than those of all other mathematical olympiads.

In 1961, teams from regions and Union Republics were invited to attend the 24-th Moscow Olympiad and so the unified multistage mathematical olympiad was began on the national scale. The first National Mathematical Olympiad was held on April 16, 1967 in Tbilisi. Since the second set of the 30-th Moscow Olympiad took place on the same day, the Moscow team had to be made up on the basis of the results of the first set of problems. Later on, the periods of Moscow Olympiads were shifted back from April to March (and sometimes to February) in order to have time to select a team for the National Olympiad held in mid-April.

At first, the Moscow team was selected directly from the results of a Moscow Olympiad. Later on, additional “qualification sets” were arranged where 15 to 20 people were invited, including those who had been awarded first, second and sometimes third prizes. The qualification problems were selected, as a rule, out of those which had been considered for the olympiad but were rejected as too difficult or because of unclear formulation (it is much easier to make a problem clearer for 15 participants in the qualification contest than at an olympiad with many hundreds of participating teenagers of various levels), or simply because they were too many. But the qualification contest has never been regarded as an additional (final) set of an olympiad. Certain problems from these competitions are given in *Selected problems*.

H.9. An accident that cased an important innovation. Once, an additional set was held: at the 33-rd Moscow Olympiad for 7-graders (1970). It was called “Pythagoras’ Day”. That year a disaster happened: The VC lost the briefcase with all papers of the 7-th grade. So the organizing committee decided to run another set. But it deviated from the established tradition. First, the teenagers were given three problems. Two hours later their papers were collected and a break for half an hour was announced after which three more problems were given.

Regrettably, it is impossible to repeat this procedure at a regular olympiad since it proved to be very difficult to collect even 7-th graders after the half-hour break and what could have happened if not a hundred but several thousands schoolchildren were set loose for half an hour to run about? It sometimes happened at Olympiads that teenagers stormed the unmanned locker room leaving behind a small pile of buttons torn off in the process . . .

Another innovation — pity, this did not become a regular practice — was of greater interest: one of the problems on the Pythagorus’ Day (33.D.7.2) was suggested by the organizing committee with no definite solution known. The participants were told that it was a research problem and they should try to advance as far as possible in solving it.

H.10. Two exceptional first prizes. There have been occasions when the first prize (no less!) was given to those who had not completely solve any (!) problem.

- At the 9-th Olympiad. Erik Balash, a 10-th grader, spent the entire time of the Olympiad trying to solve just one problem (9.2.9-10.2): *For the Fibonacci sequence 0, 1, 1, 2, 3, 5, 8, . . . find whether among the first 100 000 001 terms of the series there is a number ending with four zeros.*

The organizers thought that the students would try to solve this problem by means of relatively simple considerations related to Dirichlet’s principle. But Balash approached the problem from quite a different

angle. He decided to give a full investigation, i.e., to indicate the numbers *of all terms in the series*, which end in four zeros. For this purpose, he conducted an arithmetic investigations which he failed (or had no time) to complete. Erik pointed out correctly that the first term ending in four zeros is the one numbered 7501 and found the law of recurrence of such terms further on. The solution was marked ($\pm!$) and Balash got the first prize although he did not even start to solve the remaining problems.

- At the 8-th Olympiad. The organizers believed that the following Problem 8.2.7-8.4 was relatively easy.

Vertices A , B , and C of triangle ABC are connected with points A' , B' , and C' lying on the opposite sides, but not in the vertices, see Fig. H2. Prove that the midpoints segments AA' , BB' , and CC' do not lie on the same straight line.

FIGURE 2. (Sol. AH2)

Indeed, the midpoints M , N , and P of the pointing out segments belong to the midlines DE , EF , and FD of ABC . Hence, the assertion required, since *no line passing through the vertices of the triangle can cross all the sides in their inner points*.

The organizers believed the statement italicized is obvious. However, a participant, Yulik Dobrushin from the 8-th grade (now the world-famous mathematician, Roland Lvovich Dobrushin, Dr.Sc.), reached this stage in the solution and added: “For a long time I have tried to prove that a straight line cannot cross all three sides of a triangle at their inner points but failed to do so. I am horrified to realize that I do not know what a straight line is!”

Dobrushin was crowned with the first prize for this frank recognition of his failure. The members of the organizing committee might have understood the meaning of Dobrushin’s phrase better than its author himself. The point is that in modern geometry the answer to the question *what is a straight line* is given only by listing the line’s properties among which the impossibility to cross all three sides of a triangle (or an equivalent property) is usually included.

H.11. The rise and fall of the Olympiads. The main mathematical forces in the USSR had been concentrated at the “mekh-mat” of Moscow University and at the V. A. Steklov Institute of Mathematics of the USSR Academy of Sciences until the 1960s. Later on many young mathematicians appeared also in other educational and research institutes¹. They were very enthusiastic about preparing problems and holding olympiads: to preserve the spirit of science. Some institutions of higher education in bigger towns started to arrange their own olympiads and, in addition, the level of district olympiads was raised.

It was decided that Moscow Olympiads should (1) be held by Moscow University jointly with MGPI (Moscow Federal Teachers Training Institute) and the Moscow Institute of Railway Engineers (MIIT)², and (2) have only one set for junior grades.

The grades were divided among the Institutes. The Moscow University was to hold the olympiads for the 7-th and 10-th grades while MGPI took the 8-th graders and MIIT the 9-th graders.

The olympiads for the 7-th grade has been conducted by the Department of Computational Mathematics and Cybernetics of Moscow University since 1981 (and now it runs the olympiads also for 9-th graders). The organizing committee meets to discuss problems, to sum up an olympiad and, for other matters held jointly, at the Moscow University, as a rule. The review of the results and the awarding ceremony also took place there.

The first set of the 37-th Olympiad (1974) was held only for pupils of grades 9 and 10 while at the 38-th to 40-th Olympiads it was provided only for 10-th graders. The results of the first set were taken into account in the general review and the participants who had solved the problems of the first set received a prize or a certificate of merit one degree higher than they would have been entitled to simply from the results of the second set.

¹Thanks to the state antisemitism: they were not admitted to the principal Universities.

²This was done in 1975.

Starting with the 41-st Olympiad (1978), it was decided to hold only **one** (final) set for all school grades since the role of the first set was played by the district mathematical olympiads whose winners were allowed to participate in the final of the Moscow Olympiad.

We will not dwell here on matters pertaining to the work of the organizing committee. Suffice it to say that the holding of an olympiad, publication of the collection of preparatory problems, compilation of a long list of new problems and selection of variants of the olympiad's problems requires a tremendous effort whose magnitude the majority of the participants do not even approximately appreciate and which would have been impossible without enthusiasm of, mostly undergraduate, mathematics-major students and post-graduates and without help from the Moscow Mathematical Society and other organizations.

When an olympiad is held, the working day of the organizing committee members may last from early morning till late at night. The papers are checked and rechecked several times and quite a few teenagers got their certificates due to attentive members of the organizing committee, who were not lazy to reread carefully the solutions to find the grain of rationality in them.

The teachers and school pupils most often ask if a pupil who did not attend the circle has any chance to win an Olympiad?

There is only one answer to this question: yes, of course one has. (But the point is that this question is a "wrong" one: we would like to teach to value mathematics, rather than the accompanying sports.) Among the winners there always were kids who had not been members of any mathematical circle. Many of the participants and winners of an Olympiad came to the circle on the subsequent academic year and then took part in Olympiads (or, sometimes, willingly refrained from participating; having made a choice between sport and science). Of course, the systematic studies in the circle, the mathematical culture and skills in solving mathematical problems acquired there came in very handy for participation in an Olympiad.

While the circle involved several hundreds of Moscow teenagers in systematic work, the number of participants in a Moscow Olympiad was always considerably greater and was as high as several thousands. For example, in 1964 there were over 4000 participants, the 1966 Olympiad was attended by about 5000 boys and girls while in 1974 their number reached 6000(!).

True, this figure decreased later but still the count was in thousands; a thousand school students came to the jubilee Olympiad in 1985. all rooms at the University were overcrowded in those years and some of the participants had to be placed in laboratories of the physical, chemical and biological departments.

H.11. Tournament of towns.

H.12. On relation of olympiad problems with the "big" mathematics. As for the olympiad problems, there are stringent requirements: the problems should be diverse in form and in ideas they are based on but their solution should not go beyond the limits of the existing school curriculum. Two in five or six problems are generally simple; algebraic and text problems alternate with geometric ones while their complexity usually grows as their number increases in the assignment list.

Notice that the problems given at olympiads are non-standard. Their novelty and attractiveness can be explained to a great extent by the fact that they are inspired by fresh ideas of modern mathematics and *every one of them* is a small investigation opening up new horizons for the person who tries to solve it. Quite a few olympiad problems are related to "serious" mathematics. Here are some examples of such problems:

Problem **9.2.7-8.4** came from crystallography and is related to growth of crystals. When crystals start to grow in a solution, a crystal stops growing if it comes up against another crystal (in the problem "a car finds itself in front of a road block").

Problems **9.2.7-8.5** and **9.2.9-10.5** are related to the theory of projective planes over finite fields.

Problem **13.1.9-10.1** was taken from the "*Imaginary Geometry*", the famous book by N. I. Lobachevsky, one of the discoverers of the non-Euclidean geometry and the one who described its *theory*.

Problem **15.2.10.1** is associated with *Lagrange's problem* in celestial mechanics.

The concept of an "attracting" point and a "repulsing" point in the *iteration method* was reflected in Problem **20.2.10.2** (all equations of the problem have the same form "iterating", so to say, the first equation).

The solutions of Problems **21.1.10.5** and **31.2.7.5** use the concept of the *world line* in time and space.

26.2.8.1 is a problem on *Young tableaux* used in the representation theory of symmetric groups.

29.2.8.3 and **29.2.9-10.3** are typical problems of the *information theory*.

Problem **30.2.10.1** is "the exchange transformation" from *ergodic theory*.

31.1.9.4 is the first problem in the *coding theory* ("the check for evenness" by Hamming); cf. also Problem **30.1.8.3**.

The question raised in Problem **47.10.2** is related to the one of the ways of *tight packing of information* in computer memory while Problems **45.7.1**, **45.10.3** and **48.10.3** are indirectly connected with the *theory of algorithms and computations*.

Problem **48.9.5** was taken from the note-books of one of the greatest mathematician, Leonard Euler, and is connected with the *ideal theory*.

The fast speed of convergence to a fixed point (to $\sqrt{2}$) in Problem **49.8.3** was occasioned in the general case by *Newton's method* for finding roots of the arbitrary function $f(x)$.

Problem **49.10.5** is related to the theory of approximations of functions.

The list of examples can be extended further (e.g., to indicate some problems from the number theory) but unfortunately it is impossible to explain the idea in more detail if we want to remain on the high school level.

The school curricula have been changed several times for the last 50 years and new trends in the curricula immediately affected problems of Moscow Olympiads. So in certain years there were given problems on complex numbers, problems with a derivative, etc. Our solutions and hints correspond to the present school curriculum although it is worth saying that a different (often more cumbersome for the lack of an adequate language) solution was expected from the participants in some cases. For

example, in Problem 15.2.10.1 we made use of the properties of the integral, well-known to today's school pupils, and gave a solution taking up just a few lines (in contrast to the two-page solution of this problem in [SCY].)

H.12. What makes Olympiads run. Despite the great help of enthusiasts, the compilation and selection of an olympiad's problems is one of the most arduous tasks in the work of the organizing committee. It is the subject for debate at a number of meetings that last for many hours and where the organizing committee members argue till they are blue in the face fighting for some problems and rejecting others. Problems may change beyond recognition before one's eyes; sometimes several seemingly quite different ideas are integrated into one problem but sometimes, on the contrary, one problem disintegrates into two or three others that may be from different mathematical disciplines.

When problems are selected for an olympiad, they have to be kept in secret, on the one hand, but, on the other hand, attempts are made to find out whether heads of circles (who never are members of the organizing committee) have ever given the same or similar problem to their disciples. This was always a delicate matter since a problem unknown before could be made public by chance and so spread widely among students. So the final selection of problems have always been quite difficult.

As the olympiad approaches, the 'problem rush' increases more and more. More often than not, the final list of problems is approved one or two days before the start but it has also happened that the list was typed during the night on the eve of an olympiad. So one should not blame the organizing committees of olympiads for a rushed work in this respect since it was the compilation of final variants literally on the eve of contests that made it possible to keep the problems in secret and also to take advantage of lucky discoveries made, as usual, at the last moment.

The complexity of problems at olympiads varied noticeably in different years. The most difficult problems of early olympiads which were solved by just a few participants, now look nothing out of the ordinary. The inquisitive reader will notice that the style itself of later problems has changed substantially as compared with that of the first olympiads. However, the complexity of the olympiad in each particular year has always been very high. Sometimes it was impossible to make variants easier however hard the authors worked on it. There have been some particularly difficult olympiads, including the 27-th (1964), 29-th (1966), 31-st (1968), and 35-th (1972). Nobody solved some of the problems in these olympiads (we can cite as an example Problems 29.2.8.5, 31.2.8.2, 31.2.8.3, and 35.2.9.3) and sometimes only one participant succeeded (e.g., Problem 35.2.9.1).

The jubilee 48-th Olympiad (50 years of Olympiads) can not be called very difficult; still, it had problems that none of the kids in the respective grade could solve (48.7.4, 48.8.5, and 48.9.5). But all this is exception rather than a rule; every problem at most olympiads was solved by at least one participant and there were difficult problems solved by many participants.

The spirit and nature of an olympiad, and the content and complexity of its problems were affected to a great extent by the professors of mekh-mat, who headed the organizing committee in different years and who were entrusted with this task by the Board of the Moscow Mathematical Society.

Here is the list of the Chairpersons of the Organizing Committee:

Olympiad	Year	Chairperson	Olympiad	Year	Chairperson
1	1935	P. S. Alexandrov	23	1960	I. R. Shafarevich
2	1936	N. A. Glagolev	24	1961	V. A. Efremovich
3	1937	A. N. Kolmogorov	25	1962	N. V. Efimov
4	1938	A. G. Kurosh	26	1963	A. N. Kolmogorov
5	1939	L. A. Lusternik	27	1964	I. R. Shafarevich
6	1940	L. S. Pontryagin	28	1965	N. V. Efimov
7	1941	A. O. Gelfond	29	1966	A. A. Kronrod
8	1945	I. M. Gelfand	30	1967	V. V. Nemytsky
9	1946	S. A. Galpern	31	1968	N. S. Bakhvalov
10	1947	I. G. Petrovsky	32	1969	V. A. Efremovich
11	1948	V. V. Nemytsky	33	1970	V. M. Alekseev
12	1949	A. I. Markushevich	34	1971	I. R. Shafarevich
13	1950	M. A. Kreines	35	1972	B. P. Demidovich
14	1951	B. N. Delone	36	1973	A. A. Kirillov
15	1952	P. K. Rashevsky	37	1974	V. I. Arnold
16	1953	D. E. Menshov	38	1975	A. N. Kolmogorov
17	1954	S. V. Bakhvalov	39	1976	A. V. Arkhangelsky
18	1955	G. E. Shilov	40	1977	V. A. Uspensky
19	1956	E. B. Dynkin	41	1978	Yu. I. Manin
20	1957	O. A. Oleinik	42	1979	V. M. Tikhomirov
21	1958	V. G. Boltyansky	43	1980	A. S. Mishchenko
22	1959	E. M. Landis	44	1981-??	O. B. Lupanov

After 1981, the prime of "stagnation period", and by inertia after it the Chairman was the Dean of mekh-mat, Prof. O. B. Lupanov.

The Chairpersons of the organizing committee usually did not overwork. The Vice-Chair, on the other hand, not only worked hard to arrange the event but also headed the numerous meetings of their organizing committees, etc.

The VC had several main assistants, including heads of school grades. Among VC's numerous responsibilities were arrangements for lectures to be delivered to the school students, putting up posters of the olympiad, the run of the olympiad itself, publication of the collection of preparatory problems and blank forms for certificates to be awarded to winners, arrangements for the rooms where the olympiad is to be held, to be followed by reviews of problems and the awarding ceremony, etc. In addition, the VC had to participate in discussions of problems and help to conceive them. One of the authors of this book served as VC and can testify how difficult this job is.

The grade's managers are also very busy; in addition to helping the VC in organization, they must also select the problems they like best for their respective grades to be used later by the organizing committee in the final variant of problems for a particular school grade. They should also supervise the progress of the olympiad in their grade, go round the rooms and answer

questions of participants (to an extent specified by the organizing committee in advance); organize the checking of the papers in their grades, and find the best papers presenting all its decisions to regular meetings of the organizing committee.

When an olympiad is in progress, it is served by many university students who help the school pupils to find their rooms, sit in the rooms (answering questions if necessary to an extent specified by the organizing committee in advance), see to it that the corridors and toilets are not turned into discussion clubs by the participants, and collect papers at the end. Then heads of grades distribute the collected papers among the university students for checking and grading.

H.13. How grading is being done. The grading is the most important part of the job; it often happens that an interesting paper is read by several members of the organizing committee and those nominated for an award by all members. The diversity of demands placed on papers makes the opinion of each individual member of the organizing committee rather subjective and it is only the collective discussion makes the final decision correct and objective.

The papers are evaluated not in marks as at school but using a more flexible system of pluses and minuses. The marks that a solution may get are:

- (0) there was no attempt to solve the problem;
- (−) the problem was not solved or solved incorrectly;
- (−?) the solution is wrong and contains very bad errors;
- ($\bar{\quad}$) the problem was not solved but there are some reasonable considerations in the draft or in the clean copy;
- (\mp) the problem was solved incompletely but the approach is correct;
- (\pm) the problem was not solved completely;
- ($\overset{+}{-}$) the problem was solved but the solution contains small omissions or defects;
- ($\overset{-}{+}$) the problem was solved completely;
- (!) the solution contains unexpected (and sometimes even unforeseen by the organizing committee) bright ideas.

Other marks are also used sometimes (e.g., $1/2$, ε , etc.). It should be noted that the mark (!) does not mean that the problem was solved. For example, there are often marks $-!$, $\cdot!$, $\mp!$, etc. However, even $(-!)$ increases considerably the chance to get a prize.

Everything is taken into account when prizes are awarded, including the correctness of a solution, the clearness of the mathematical thought, originality of the solution, the completeness and fullness of the investigation, the nature of the description of fine points, etc. However, the handwriting and the tidiness in the arrangement of the material, as well as the general appearance of a paper are never considered, unlike the regular procedure in a usual school. The greatest importance is attached to non-standard reasoning, unexpected solutions, and the original interpretation of the conditions of a problem.

H.14. Anecdotes from the history of Moscow Mathematical Olympiads¹. Olympiad is a great event for schoolchildren who are interested in mathematics. The faculty members and the students at the Department of Mechanics and Mathematics of Moscow University are barely able to cope with the multitude of questions fired at them by excited teenagers and sometimes by no less excited teachers:

“When will lectures be provided for participants in the Olympiad?”

“Are any consultations planned?”

“Is it only for the best pupils or for all?”

“Where can we get problems for practicing and how many of them suffice to be solved?”

“Will a boy be permitted to participate if he is only from the 6-th grade?”

“Can we bring textbooks with us?”

The stream of similar questions never stops.

- The desire to be as objective as possible and the great awareness of the organizing committee members of their duty sometimes resulted in curious situations. Consider just one such case. At the 21st Olympiad, a paper by Misha Khazen who had solved four problems out of five was nominated for the first prize. Unfortunately for him, Lida Khazen, Misha’s sister, was a member of the organizing committee. She stated with assurance that Misha had known a solution of one problem before the Olympiad (although he solved it himself), that he was not going to apply to the Department of Mechanics and Mathematics anyway, and so he should not be awarded the first prize. The members of the organizing committee spent a lot of time trying to prove to Lida that the inclusion of a known problem into the Olympiad was the fault of the organizers but not Misha’s, that the accidental relationship of Misha with one of the organizing committee members allowing them to learn what he knew and what he did not know put him in more difficult conditions as compared with the others, that the question of entering the University was of no importance, that, in general, they discussed the paper but not its author, and so on. Nothing helped. The poor girl was on the verge of crying and only the democratic procedure of voting (perhaps an hour or an hour and half after the debate has started!) made Lida agree.

- For a long while it was a custom to include in the final list of problems the one whose answer is the year of the current Olympiad. The following solution of one of the participants of the 33-rd Olympiad put an end to it: “At every Olympiad there was a problem whose answer was the year the Olympiad was held. In this Olympiad the problem I am solving is the only such problem. By the induction, the answer: 1970”. (Cf. Problem 33.2.7.4.)

H.15. Who did what (very incomplete). Several generations of outstanding mathematicians have worked on the main material of this book — the problems — and some of the problems are really nice. To find the authors of most of the problems is impossible. Besides, part of interesting problems are results of brainstorming held at the meetings of the organizing committee, and so they have a collective author.

However, the most beautiful (in our opinion) and original problems were devised by individual authors and we are sorry that can not mention all of them. Such problems were widely spread first among the organizing committee, and, after the olympiad, became a mathematical folklore. Experts recognize them at once by their nicknames. Here are some authors of such problems (this list is incomplete in every sense; we hope that the authors not mentioned will not be offended):

N. N. Konstantinov: 17.2.7.5 (Triangular City) and 23.2.8.4 (Snail); S. A. Eliseev: 38.2.9.5 (Non-convex Cutting); D. B. Fuchs: 24.1.8.2 (Scalar Product), 27.2.11.5 and 31.2.9.2 (Fuchs’ Arcs); G. A. Galperin: 33.2.7.6, 34.2.7.5, 38.2.10.4,

¹Some of these stories might sound strange for the Westerner, more used (and sometimes prone) to esteem the law.

33.2.7.6 (Courtiers of King Louis), 38.2.7-10.1 (The 2s, 3s and 4s), 39.2.9.5 (Nickels) and 48.9.2 (The Airfields); S. B. Gashkov: 46.10.3 (The Turing Machine), 48.7.4 (Wolves and a Hare) and 48.10.3 (Complexity); B. D. Ginzburg: 23.2.9.3 (Knight's Way); A. I. Gruntal: 36.1.10.4 (The Polyhedron); V. Gurchich (a complete graph); V. G. Kac: 30.2.9.5 (A King of Spain); M. P. Kovtun: 34.10.5 (Matches); A. V. Klimov: 37.2.10.3 (Asteroids); S. V. Konyagin: 42.8-10.5 (Chemists & Alchemists); A. G. Kushnirenko: 33.2.10.1 (The Orange); O. V. Lyashko: 35.2.7.4 (The Knight-Errand); A. P. Savin: 33.1.9.1 (Extreme Kings); I. N. Sergeev: 48.8.4 (Uncle Chernomor²); A. S. Shvarts: 24.2.9.2 (Shvarts' Matrix); A. C. Tolpygo: 30.2.7.3 ($q \cdot 2^{1000}$), 30.2.8.5 (moovies), 31.1.9.4 (telephones), 32.2.7.3 (a trickster) and 32.2.10.1 (wizards); A. L. Toom: 35.1.9.3 (The Forest); N. B. Vasilyev: 25.2.10.4 (The Box) and 26.2.8.5 (Gentlemen); A. V. Zelevinsky: 34.2.10.3 (Banker and Gambler).

One of the problems of the Pythagorus' Day (33.D.7.5), as it turned out, suddenly became quite popular outside the USSR. "Mathematical Gardner" [Kl] contains its generalization for the case of a "many-handed Ali-Baba" given in the section entitled "Entertaining Table-Turning". It said there that the problem visited first the pages of Scientific American in 1979, where it was published by Martin Gardner, a famous popularizer of mathematics, well-known to Soviet readers from a number of books and articles (see refs. [G1]–[G14]). However, Gardner admitted that he had got this problem from "Robert Tappey who believed that the problem had come to us from the Soviet Union." ([Kl]). Thus, Problem 33.D.7.5 has come a long way before returning home (anonymously), albeit in a generalized form.

The authors of the latest Olympiads, held in the copyright era, were mentioned explicitly on various liflets issued on the occasion, so we dutifully reproduce the information.

The list of authors is easy to extend but almost impossible to complete (let alone the fact that the above mentioned authors suggested far more problems from this book than we mentioned); some authors donated many problems without bothering for stacke claim (like Joseph Bernstein, who in his time solved all problems offered in Olympiads he participated). We apologize to all authors of problems for Moscow Mathematical Olympiads who are not mentioned.

²The Russian reformulation after a fairy-tale hero.

A little problem

I. Zverev

To Mark Scheinberg, a much honored student of the 9-th grade and the winner of many a mathematical olympiad, with the highest respect from the author who can hardly remember the multiplication table.

“Well, that’s it,” Leo said. “Settled and signed.”

“Who by?” Mashka asked.

“By myself,” said Leo solidly and looked at her in a severe schoolteacher’s way, his eyes like goldfishes behind his glasses. “Not enough for you?”

“And also by me,” Yura Fonarev added.

“O.K., count me in then,” Mashka sighed. “I agree.”

“What do you mean ‘you agree’? Nobody’s forcing you.” The boys looked at her with indignation.

“It would be too much if you forced me.” Now Mashka got angry. “Too much, really.”

“There, there,” Leo said soothingly. “You won’t regret it. You’ll be grateful. Do you know what kind of school it is?”

Mashka knew no less than they did. They had been there together on the Open-Day.

A week earlier, Ádochka (this meant Ariádna Nikoláyevna, their math teach), had informed Leo, their class genius, that there would be such an occasion and so she would advise him to

The three of them went together. The school was really amazing. The classrooms were not called classrooms but *auditoriums*, one of them even had a computer¹. The lessons were called *lectures* and they were given not by schoolteachers but by professors from the University, among them even one full professor with the doctoral degree.

Of course, the students there might have stretched a bit but prodigies there really seemed to fulfill the freshmen’s and partially sophomore’s curriculum in the 9-th grade. So it would have been ridiculous for those who were not straight A students to even think of getting in.

Yura had two B’s, Leo had no B’s, but he had an uncomfortable C in German. Mashka, of course, had lots of B’s.

When the Open-Day was almost over and the boys were about to leave, there appeared the principal of this special mathematical school. He was a rather strange man, cross-eyed, with a big nose and wild hair, just like Leo’s but grey. He said right away that those who were not entirely straight A students, but talented nevertheless, shouldn’t give up; for it was the results of the Mathematical Olympiad that would count most of all.

It was, therefore, settled and signed that all three should go to the Olympiad on Sunday and take a chance.

Mashka didn’t think it was such a great idea to try and enter this mathematical school. She had other plans. She didn’t know exactly what, but no doubt they had nothing to do with mathematics. It wouldn’t have been fair, however, to leave the boys alone at such an important moment, so she would go and flunk, of course, but still give their morales a boost.

“Are you going too?” asked the surprised Ariadna Nikolayevna and immediately blushed. Perhaps she was afraid Mashka might get offended.

Adochka was very kind, and when she accidentally hurt somebody always suffered terribly. She started to worry and tried to soothe the offended.

“I’ll do it just to keep the boys company,” Mashka comforted her.

“Oh, no!” Adochka cried excitedly. “I’ve always said you are talented . . . just a bit lazy . . . But if you try and organize yourself you may gain . . . I mean, achieve”

She said nothing more because she was honest and knew well enough that Mashka could never gain anything in mathematics, let alone achieve.

Mashka’s father was also surprised. He said, “Oh, my!”

But since he held a doctorate in philosophy he felt he had to philosophize a little. So he told Mashka’s mother how wonderful it was that their daughter had chosen such a nice field of activity, where everything is simple and clear, directives are definite and not subject to frequent change.

“But in that field you have to have a regular head on your shoulders!” Mashka’s mother exclaimed and sent Mashka away to do her homework.

⁰From *The Second of April* by Ilia Zverev, Soviet Pisatel Publishers, 1968

¹The western reader should look at the year this had been written: at that time computers were at best discussed in the newspapers in Russia.

Leo's father got very excited when he heard about the Olympiad. He started to pace back and forth and rub his bald head that perhaps once grew the same kind of black wire-like hair that his wonderful son had now.

"Listen, Leo," he said at last, "you know, physics is somehow more promising these days. Perhaps there's rocketry physics or something?"

"So what?" Leo said condescendingly. "I, for instance, like math."

Still, Leo's father would be extremely sorry for his son to get involved in a second-rate science, or even a first-rate one, if it were not the main one.

"With your abilities," he cried, "you could . . ."

"Enter a school where they teach how to run ministries," Leo prompted gloomily. "It's hard enough to enter this one. They take only one in twenty two."

His father immediately found another subject to worry about: "What if they don't enroll you, Leo? You must go to your headmaster," he said, "and to the Young Communist League, too, and get letters of recommendation from all of them. Make them write that you are one of the best students and a member of the committee . . . and about the physics club that you are the monitor of . . ."

"Oh, God," Leo said. "And that I bought a light bulb for the physics classroom with my own 30 kopeks. That's also a feature of my character that is a visible sign¹."

"Don't show your wit here," his father ordered. "I've lived longer and I know better what plays sense in cases like this."

Leo's father was a musician. He played the trumpet and perhaps that's why he thought one could play anything, even sense. He was not too literate because he had joined an orchestra as a prodigy right after his fifth year in elementary school. Of course, now times were different. Prodigies had no privileges. On the contrary, they had to study five times as hard as all the others. Leo put all this into one sentence:

"Daddy, you are out of tune."

But after thinking it over he did decide to get the damn recommendation. It really was highly unlikely that anything of the sort would be required.

Finally, came the morning of the judgement day. That was how Yura chose to call it. For everyone else it was an easy Sunday morning but for 563 students "talented in mathematics", as they were formally called, that morning was most uncomfortable . . .

The boys crowded the wide University staircase decorated with statues of various bearded thinkers. Some of the crowd stood motionless, staring upward and silently moving their lips, perhaps praying or, much rather, solving problems. Others were nervously discussing tricks from the last Olympiad, and of the one before the last.

The girls stood separately. They were bespectacled and very serious. "Abstract", as Yura put it. One with a forelock was surprisingly cute. It was hard to understand what such a beauty needed mathematics for.

The most brave (or, more precisely, the most anxious) had the nerve to come with their parents, and now, shy and suffering, they received fatherly advice and motherly instructions.

"Most important, don't be nervous," a fat red-faced woman in a fur-trimmed coat kept saying to a fat pink-cheeked boy. "I beg you, Noughty!"

What a mathematical name, Noughty. Wonder, what his real name was? Arnold, perhaps? He was pretty nervous, that Arnold 'boychick'. He'd flunk just from fright. Well, actually everybody was rather nervous that morning. Even Yura and Leo, speaking frankly.

In the midst of this excitedly buzzing, breathing, stirring and even steaming crowd, two boys were distinctly out of place, like an iceberg. They were indifferently sitting on a step playing deadman. The older one, in glasses and ski trousers, lazily pronounced after each move: "Aha, oh, well, if you do that, we do this . . ."

"That's Guzikov," Yura whispered respectfully. "Second prize at the National Olympiad." He sighed. "Of course, he can do whatever he wants now, even play deadman."

At last, a tall young man carrying a briefcase appeared at the entrance. He made a frightening grimace and shouted:

"Welcome, friends! We are starting."

Everybody began to push one another and loudly tramped their way through the shining marble hall into a very big room. Only Leo lingered at the entrance before an enormous sheet of white drawing paper. It declared:

"STUDENTS! ADDRESS YOUR QUESTIONS TO A. KONYAGIN, ROOM 9".

Leo just had one. He went to Room 9.

A. Konyagin, the question authority, turned out to be the young giant who had just shouted, "Welcome, friends!" He again made a bestial face and said in a very kind voice:

"Please, ask. I'm listening."

Leo thought this young man could be one of the poor "antipeople" Ryasha told stories about. Ryasha was a dreamer of course, and liked to fib but this particular story sounded real. He said there were such "antipeople". Ryasha even remembered their Latin name, very impressive — "homogeneous lupusest". They could never do what they wanted. If such an antiguy, say, wanted to cry, he would laugh instead, and if he wanted to run around, he would go to bed immediately instead. Ryasha swore that it was a quite established scientific phenomenon, well-known in medicine. He might know, after all, since both his parents were doctors.

"Well, what is it?" Konyagin got angry and his face turned accordingly kind. "Speak up!"

Leo asked his question. "Do they require letters of recommendation? What other papers are needed?"

"Papers? That's where your papers are!" Konyagin knocked on his protruding forehead. "Here is your recommendation and reference, and permit. Clear? Then, go ahead!"

In the big room, called auditorium No. 1, stood twenty rows of benches. Very long benches they were, and each had a desk in front of it.

¹[of a Communist morale]. A word from mass-media cliché of that time; (like pledge of allegiance in American schools).

“Two at each desk, not more,” said the question man Konyagin and headed down the aisle. Five scientists, also young and looking very important, went after him, distributing paper.

“The sheets are stamped,” Konyagin said as he walked. “Don’t even think about cheating. No way! We are not so old here. We still remember all the tricks ourselves. Mind that!”

“So it was OK with you?” someone squeaked challenging. Perhaps it was that Noughty one, that pink boychick with the mathematical name.

“But I never cheated in math!” said the question man proudly, at which his assistants burst out laughing for some reason.

Each had his own problem. No ordinary problem about the Collective Farm “Shining Path” that bought two tractors and three vans while the Collective Farm “Dawn” acquired seven tractors, etc. No, these assignments were quite different.

Yura had one about King Arthur and his knights. Knowing that each knight was at war with half of the others, how should King Arthur’s right hand man, Sir Lancelot, arrange them around the table so that no one should sit beside his enemy?

“What have you got?” Yura asked Leo. Of course, he had to know about Leo’s problem first.

Leo had a problem about chess players. Eight chess players took part in a competition and each finished with a different score. The second best had the same score as the four worst combined. What was the score of the game between the fourth and the fifth?

Leo thrust his fingers through his wild hair and began to breathe, moo and blink. This meant he was starting to work.

“Let’s reason!” he persuaded himself aloud. “Let’s think logically and calmly. Each of these guys played with each other and either won, lost or drew. So the first one . . . But what am I doing?” Leo interrupted himself noticing that he plunged into his business while his friend might be in trouble. “So, how many knights do we have?” He said it just like that, we.

Mashka, of course, could solve nothing but she could not go away because the boys might think that she had solved her problems before them and feel uneasy. The possibility was pretty hypothetical: of course, the old friends could guess that Mashka’s poor math wouldn’t work here. Still, she was pleased to think that, sitting there, she could somehow inspire these budding Euclids and Lobachevskys.

She just shuffled her clean sheets of paper with purple official stamps and looked around at the people. There was a lot to see for a detached observer. The great Guzikov wrote his figures as if he were playing piano. He thrust his head upwards, raised his eyebrows and even jerked in rhythm to an inner music. Noughty was strangely calm. His pink face shone with satisfaction. Perhaps he had been lucky enough to draw an easy problem.

Occasionally, a boy went to Konyagin and whispered for permission to go to the bathroom.

“Leave your pen here,” said the question man to one. “You don’t need a pen in there, do you?”

As more of the boys asked to go, the assistants looked at each other meaningfully demonstrating that, of course, they knew the secret aim of those visits though the aim might not be secret but quite a natural one. After all, the Olympiad lasted five hours.

Everybody, except Mashka, was suffering, writing or thinking. The cute girl with the forelock — Mashka could swear she would solve nothing and had just come to show off her beauty to the young intellectuals — well, she was also writing and even confidently and merrily.

She could hardly make anything out by looking at Yura and Leo, though naturally she was looking at them most of all. They were whispering, looking into each other’s notes and arguing.

Unfortunately, not only Mashka saw that. Every now and then Konyagin looked at the friends and shook his head making his antiface and antismiles. The boys continued whispering, writing and whispering again. The fools obviously forgot where they were . . .

It all ended rather sadly. When Yura and Leo handed in their papers — not among the first, but far from the last — the question man gave them a fierce smile, took out an enormous red marker and slashed on every sheet.

“Mark my word, something terrible is going to happen,” said Mashka.

But the boys were filled with joy of victory. They didn’t want to listen to reason. They jumped, nudged each other, and shouted, because they had solved all their problems.

The joy reached the two families. Leo’s father was extremely happy but having regained self control he claimed that there was nothing to be glad about, it was quite natural, and he had expected nothing less from his son, whom he knew as well as he knew himself. He was much happier about the system where no papers were required, where they just said “go ahead, show what you can do, and that’s your whole file”.

Of course, his father liked this system because he had always had to write in his application forms: “Education: incomplete secondary school”, and some other things on top of that¹.

Fonarev’s father, having heard Yura’s account, silently took his wonderful *Poliot* watch (written just like that, not in the usual Cyrillic; but export make) off his wrist and gave it to his son. A quarter of a century had passed since his last arithmetic class but his horror of the science had hardly diminished. Every other year or so, Fonarev’s father had the same nightmare: his redheaded math teacher, Faina Yakovlevna, a swimming pool with two pipes, and Berezanskaya’s book of arithmetical problems.

“Yes, Yurka,” he said, “a regular guy you are, that’s it! Nothing more to add.”

Mashka didn’t want any lengthy explanations. She just told her parents that she had not gone to the Olympiad, she had merely changed her mind at the last moment, that’s all.

“But what did you do all Sunday?” Her mother was horrified.

“I was busy with my Russian,” replied Mashka.

“She means she was speaking Russian and no other language,” remarked her father sarcastically.

Two more weeks passed and again they went to the University. In the same auditorium No. 1, at the presidium table were three Academicians, the hairy principal of the special mathematical school, a representative from the municipal board of

¹In the standard questionnaire the line preceding the question on education required to state the ethnic origin which must have been a trial for him judging from his (manifestly Jewish) name and appearance.

education, and other officials. A. Konyagin was no longer in charge. He was somewhere in the seventeenth row with all the other assistants who turned out to be just graduate students helping to run the Olympiad.

It was a ceremony held to mark the results of the competition of mathematicians. The representative of the education department read a speech in which he emphasized achievements and pointed to some isolated shortcomings.

"We are also concerned," he mumbled indifferently, "about the level of education in some schools." At this point he at last looked up and said sternly: "No, comrades, we are not alarmed. But, comrades, neither are we satisfied."

And then, the chief Academician rose and handed awards to the winners. It turned out that the great Guzikov got only the second prize. So did the cutie with the forelock. The first prize went to that pink Noughty, the mother's darling. The boys thought they knew people well but they made a mistake underestimating him.

That psychological miscalculation was not their worst disappointment. The list of winners was apparently about to be exhausted and the friends hadn't been called yet. At last the chairman finished announcing the winners. He folded his sheet in two and then in two again and before leaving said:

"And, well . . . a Fonarev and, a mmm . . . Makhervax are requested to come to Room 9."

I don't like to describe what happened in Room 9. There weren't any Academy members there, only the principal of the mathematical school and a representative of the education board, the one who was neither alarmed nor satisfied. Konyagin was also there and kindly smiling started to lecture the boys:

"What shall we do with you? Whom should we give the prize? You solved all your problems properly but you were whispering all the time and we don't know which of you did what."

The boys started to explain that they had solved the problems together, they always did everything together, there was no crime in that because history is full of such cases. They recalled Pierre and Marie Curie, or, say, Lomonosov and Lavoisier, though they were not quite sure about the latter.

"Well, stop it," said the representative, "it's a matter of principle. The Olympiad was for individual work and prizes are given to individuals. We have discussed this with the comrades and decided as follows. You work out who deserves more and we'll give him the award. The other will have to pass the entrance exams on the regular basis."

"Here he is, Fonarev," Leo prompted immediately.

"Write Makhervax in," Yura shouted, regretting that Leo was the first to shout the right thing.

"There, there," said the hairy principal. "Go and think. Come back tomorrow morning with your decision."

When Leo's father heard what had happened, he declared that he wouldn't let it go just like that. To him it was a pure crime to prevent the country from having two geniuses instead of one.

"Why 'prevent'?" Leo felt bad but not bad enough to compromise with his conscience. "It's all right, I'll just have to pass the exams like everyone else."

Leo's father, who used to be a prodigy, as you might remember, didn't like this "like everyone else" at all. He said that it was not Leo's business to discuss such serious matters; he would go to Yura's parents and they would settle everything as serious adults should.

So the parents started to decide. First, they spoke about the weather, then about soccer. At last Yura's father decided that the stress was too much and he said with dignity that he knew his son well, was sure about him and that Yura would reach his goal whatever might happen.

Leo's father readily agreed that Yura was a strong personality, no obstacles could stop him, while Leo, of course, was rather unstable, no athlete, and wore glasses. So it would really be better to make it easier for Leo since Yura, as his father so rightly noted, would manage anyway.

Here Fonarev's father cried, "Oh!" because his wife had stamped on his toe under the table.

"No," she said bitterly, "our Yura only looks strong. And his marks are not so stable. Two B's, you see. But yours is doing just fine. Of course it will be easier for him."

"But he has C in German," Leo's father cried. "Do you understand, C! That's much worse than your two B's."

All three looked at each other rather ashamed. Somehow their talk was strange, even uncivilized.

"Katya, look to the kettle, please. I think it's already boiling," Fonarev's father said crossly to Fonarev's mother.

"Yes, really," sighed Leo's father. "It turns out rather unseemly."

"Yes," agreed Fonarev's father, "looks foolish. "Then he suddenly beamed and suggested: "Let's settle it fairly! Heads or tails."

"Heads or tails?" repeated Leo's father with doubt. "Huh . . . all right then, tails."

Fonarev took a coin out of his pocket, put it on his big black-rimmed nail, the coin flipped in the air several times and landed on the table.

"Heads!" shouted the lucky one. Leo's father shrugged and sighed.

"Maybe you think I cheated?!" asked Fonarev warily.

"No," Leo's father said sadly. "I didn't think that."

They didn't talk for quite a long time until the kettle, which had not been about to boil, was ready at last.

But there was no reason for them to be so sad.

Everything was wonderfully settled already. Perhaps not too wonderfully but settled nevertheless. Yura and Leo accompanied by Mashka went to Room 9 and took the award paper from Konyagin. The paper had "Fonarev" on it because Leo had managed to shout Yura's name first.

When the boys came out of the building, they tore the paper in halves. First they wanted to tear it in three, because they said Mashka had a right to a piece, but she protested. She said it was a token of their friendship, hardened in battle in which she, although a friend, hadn't taken part. Mashka said she would sew them special safe bags that they could hang around their necks and hide under their shirts on most festive occasions. The boys nodded. All three thought it was an excellent idea. It was quite proper for real knights, even for those they had helped to seat around King Arthur's table.

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