Sri Lankan Mathematics Challenge Competition 2013

April 06, 2013

Time allowed: Four and a half hours
Instructions:

- *Full written solutions are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation.Work in rough first, and then draft your final version carefully before writing up your best attempt.*
- *One complete solution will gain far more credit than several unfinished attempts. It is more important to complete a small number of problems than to try all five problems.*
- *Each problem carries 100 marks.*
- *Calculators and protractors are forbidden.*
- *Start each problem on a fresh booklet. Write on one side of paper only. On each booklet, write the number of the problem in the top left hand corner, and your name in the top right hand corner.*
- *Return all the booklets after the exam is over. Leave your rough work.*

Problems:

1. Let ABC be an acute-angled triangle with $AB < AC < BC$ inscribed in the circle Γ_1 with centre O. The bisector of $\angle CAB$ intersects BC at D and the circle Γ_1 at A and K. The circle Γ_2 has its centre on the line OA and passes through A and D. Apart from A, Γ_2 intersects AB at E and AC at F. If M and N are the midpoints of the segments CF and BE respectively, prove that the lines EF, DM and KC are concurrent at a point T, that the lines EF, DN and KB are concurrent at a point S and that OK is perpendicular to TS.

2. In a 2013×2013 square, the rows and columns are numbered 1 to 2013 starting from the top left hand corner. From this the unit squares situated on both odd numbered rows and odd numbered columns are removed. Determine the minimum number of rectangular tiles needed to cover the remaining region entirely without overlap.

3. Let ABC be an acute-angled triangle. The feet of the altitudes from A, B and C are D, E and F respectively. Prove that $DE + DF \leq BC$. Determine the triangles for which the equality holds.

4. A set of n integers is called a *complete set of residues modulo* n if each of them leave out a different remainder when divided by n. Find with proof all the integers n having the following property:

   There exists a permutation $p_1, \ldots, p_n$ of integers $1, \ldots, n$ such that both the sets $\{p_i + i | 1 \leq i \leq n\}$ and $\{p_i - i | 1 \leq i \leq n \}$ are *complete sets of residues modulo* n.

5. Suppose a polynomial $P(x) = a_0 + a_1 x + ... + a_n x^n$ has integer coefficients. We say that $x_1, \ldots, x_k$ is a $k-$*cycle* if these k numbers are distinct, $P(x_i) = x_{i+1}$ for $i = 1, \ldots, k$ and $P(x_k) = x_1$. Show that if P has a $k-$*cycle* containing only integers then $k = 1$ or $k = 2$.